Preliminaries to the short course in Stable Homological Algebra

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1 The stable category of a ring

Let Λ be a ring and Λ-Mod the category of (left) Λ-modules. For Λ-modules M and N, define $\mathrm{P}(M,N)$ as the set of all homomorphisms that can be factored through a projective module.

Exercise 1. Prove that $\mathrm{P}(M,N)$ is a subgroup of $\mathrm{Hom}_\Lambda(M,N)$.

Exercise 2. Prove that $\mathrm{P}(M,N)$ is an ideal in Λ-Mod, i.e. for any $f \in \mathrm{P}(M,N)$ the morphisms gf and fh are also in $\mathrm{P}(M,N)$ whenever they make sense.

Definition 1. The stable category Λ-$\underline{\mathrm{Mod}}$ of Λ has the same objects as Λ-Mod. Define the morphisms $\underline{(M,N)}$ from M to N by

$$\underline{(M,N)} := \mathrm{Hom}_\Lambda(M,N)/\,\mathrm{P}(M,N)$$

Define the composition of classes as the class of the composition of representatives, and define the identity morphisms as the classes of the identity homomorphisms.

Exercise 3. Check that Λ-$\underline{\mathrm{Mod}}$ is indeed a category.

Exercise 4. Prove that the biproduct in Λ-Mod induces a biproduct in Λ-$\underline{\mathrm{Mod}}$, which makes Λ-$\underline{\mathrm{Mod}}$ an additive category.

Exercise 5. Let $Q : \Lambda\text{-Mod} \longrightarrow \Lambda\text{-Mod}$ be a quotient functor, sending an object to itself, and a morphism to its class. Prove that Q is an additive functor.

Notation. Let $\underline{f}$ denote the class in Λ-$\underline{\mathrm{Mod}}$ of a homomorphism f.

Now let's look at Λ-$\underline{\text{Mod}}$ in more detail.

Question 1. Given a homomorphism f, when is $\underline{f}$ a zero morphism in Λ-$\underline{\text{Mod}}$?

Answer. Precisely when f factors through a projective module. This is what the definition says.

Question 2. Given a module M, when is M isomorphic to a zero object in Λ-$\underline{\text{Mod}}$?

Answer. Precisely when M is projective. Indeed, M is isomorphic to a zero object in the stable category if and only if $\underline{1_M} = \underline{0_M}$. If M is projective, then clearly $\underline{1_M} = \underline{0_M}$. Conversely, if $\underline{1_M} = \underline{0_M}$, then the identity map on M factors through a projective, which makes M a direct summand of a projective. Hence M is projective.

Question 3. Given a homomorphism $f : M \longrightarrow N$, when is $\underline{f}$ an isomorphism in Λ-$\underline{\text{Mod}}$?

Answer (Heller)**.** If and only if there are projective modules P and Q and an isomorphism $M \oplus P \longrightarrow N \oplus Q$ in Λ-Mod with matrix $\begin{bmatrix} f & * \\ * & * \end{bmatrix}$.

Proof. The "if" part follows easily from the answer to Question 2. Now suppose $\underline{f}$ is an isomorphism. Then there is a homomorphism $g : N \longrightarrow M$ such that $\underline{f}\,\underline{g} = \underline{1_N}$ and $\underline{g}\,\underline{f} = \underline{1_M}$. The first equality means that there is a projective P and homomorphisms $h : N \longrightarrow P$ and $k : P \longrightarrow N$ such that $fg + kh = 1_N$, i.e. the homomorphism $M \oplus P \xrightarrow{f \perp k} N$ is a retraction with splitting $N \xrightarrow{g \top h} M \oplus P$. Thus we have a split exact sequence

$$0 \longrightarrow Q \longrightarrow M \oplus P \xrightarrow{f \perp k} N \longrightarrow 0,$$

and we would be done if we show that Q is projective. Apply the additive functor $\underline{(-, Q)}$ to this sequence. Since additive functors preserve split exact sequences, the resulting sequence

$$0 \longrightarrow \underline{(N, Q)} \longrightarrow \underline{(M \oplus P, Q)} \longrightarrow \underline{(Q, Q)} \longrightarrow 0$$

is still exact (in the category of abelian groups). Since $\underline{f}$ is an isomorphism and P is a zero object in the stable category, the map $\underline{(N, Q)} \longrightarrow \underline{(M \oplus P, Q)}$ is an isomorphism, which implies that $\underline{(Q, Q)} = 0$, i.e. $\underline{1_Q} = \underline{0_Q}$, and therefore Q is projective. $\square$

Next, we shall see that for any module M, the isomorphism type of the syzygy module ΩM of M in the stable category is uniquely determined. First, we need

Lemma 1 (Schanuel)**.** Let $0 \longrightarrow \Omega M \longrightarrow P \xrightarrow{\pi} M \longrightarrow 0$ and $0 \longrightarrow \Omega' M \longrightarrow P' \xrightarrow{\pi'} M \longrightarrow 0$ be exact sequences with P and P' projective. Then $\Omega M \oplus P' \simeq \Omega' M \oplus P$.

Proof. Take the pull-back of (π, π'):

$$\begin{array}{ccccccccc}
 & & & & \Omega' M & = & \Omega' M & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega M & \longrightarrow & X & \longrightarrow & P' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle \pi'} & & \\
0 & \longrightarrow & \Omega M & \longrightarrow & P & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Then both the middle row and the middle column split. Thus $\Omega M \oplus P' \simeq X \simeq \Omega' M \oplus P$ □

Corollary 2. ΩM is isomorphic to $\Omega' M$ in the stable category. □

Our next goal is to show that the operation Ω gives rise to an endofunctor on Λ-$\underline{\text{Mod}}$. For each Λ-module M, choose and fix a short exact sequence $0 \longrightarrow \Omega M \longrightarrow P_M \longrightarrow M \longrightarrow 0$ with P_M projective. Also, for each homomorphism $f : M \longrightarrow N$, choose and fix $f_0 : P_M \longrightarrow P_N$ and $\Omega f : \Omega M \longrightarrow \Omega N$ making the diagram

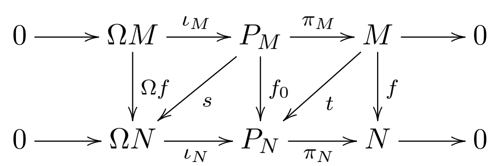
$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega M & \xrightarrow{\iota_M} & P_M & \xrightarrow{\pi_M} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Omega f} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & \Omega N & \xrightarrow{\iota_N} & P_N & \xrightarrow{\pi_N} & N & \longrightarrow & 0
\end{array}$$

commute.

Exercise 6. Suppose $f : M \longrightarrow N$ factors through a projective and $g : P \longrightarrow N$ is an epimorphism with P projective. Show that f lifts over g.

Lemma 3. If $f : M \longrightarrow N$ factors through a projective, then $\underline{\Omega f} = 0$.

Proof. By Ex. **??**, f lifts over the epimorphism $P_N \xrightarrow{\pi_N} N$: $f = \pi_N t$. By the universal property of the kernel, there is $s : P_M \longrightarrow \Omega N$ such that $f_0 - t\pi_M = \iota_N s$:



Now $\iota_N s \iota_M = (f_0 - t\pi_M)\iota_M = f_0 \iota_M = \iota_N \Omega f$, and, since ι_N is a monomorphism, $\Omega f = s\iota_M$, showing that Ωf factors through a projective. □

Lemma 4. For any homomorphism $f : M \longrightarrow N$, $\underline{\Omega f}$ does not depend on the choice of f_0.

Proof. Take two liftings of f and look at their difference. This is a lifting of the zero map. By the preceding lemma the classes of the two liftings coincide. □

Combining the last two lemmas, we have

Corollary 5. $\Omega : (\underline{M, N}) \longrightarrow (\underline{\Omega M, \Omega N}) : \underline{f} \mapsto \underline{\Omega f}$ is a well-defined map. □

Proposition 6. With the above fixed choices of resolutions and liftings, Ω is an endofunctor on Λ-$\underline{\mathrm{Mod}}$.

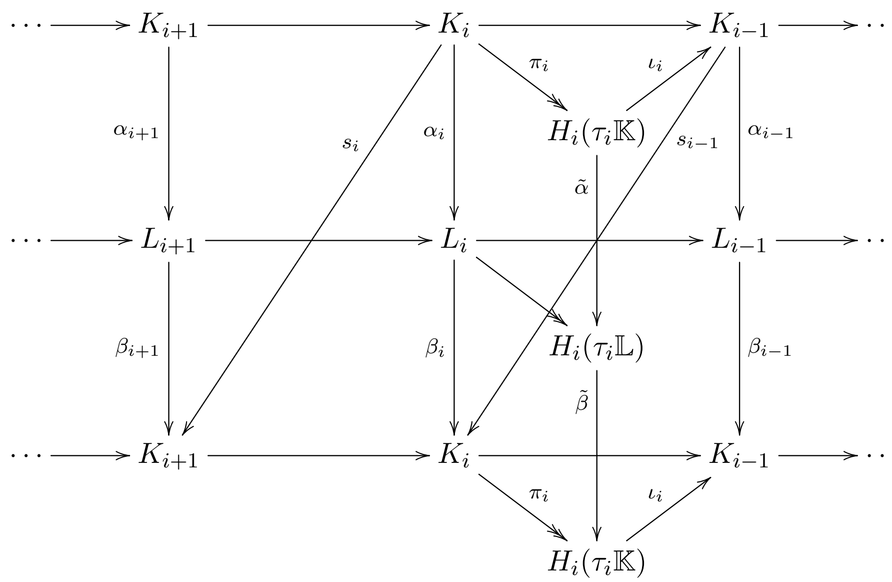
Proof. First, we show that Ω preserves the identity maps. Indeed, since both $1_{\Omega M}$ and $\Omega(1_M)$ are liftings of 1_M, we have, by Lemma **??**, that $\underline{1_{\Omega M}} = \underline{\Omega(1_M)}$, the latter being $\Omega(\underline{1_M})$ by definition (see Corollary **??**).

Next, we need to show that Ω preserves composition. But, for any composition fg, $\Omega(fg)$ and $\Omega(f)\Omega(g)$ are both liftings of fg. By Lemma **??**, $\underline{\Omega(fg)} = \underline{\Omega(f)\Omega(g)}$, the latter being $\underline{\Omega(f)}\ \underline{\Omega(g)}$ by the definition of the composition in Λ-$\underline{\mathrm{Mod}}$. □

We will need another result,

Lemma 7. Let $\mathbb{K}$ and $\mathbb{L}$ be homotopy equivalent complexes of projectives, and $\tau_i\mathbb{K}$ and $\tau_i\mathbb{L}$ the truncations at degree i. Then $H_i(\tau_i\mathbb{K})$ and $H_i(\tau_i\mathbb{L})$ are isomorphic in the stable category.

Proof. Let $\mathbb{K} \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathbb{L}$ be mutually inverse homotopy isomorphisms. Then $1_{\mathbb{K}} - \beta\alpha$ is 0-homotopic, with homotopy, say, s. We then have a diagram



where $\tilde{\alpha}$ and $\tilde{\beta}$ are induced by (α_i, α_{i+1}) and (β_i, β_{i+1}). Notice that $(1 - \tilde{\beta}\tilde{\alpha})\pi_i = (\pi_i s_{i-1}\iota_i)\pi_i$. Since π_i is an epimorphism, $1 - \tilde{\beta}\tilde{\alpha} = \pi_i s_{i-1}\iota_i$, and therefore $\underline{1} = \underline{\tilde{\beta}}\ \underline{\tilde{\alpha}}$ in the stable category. Likewise, $1 = \underline{\tilde{\beta}}\ \underline{\tilde{\alpha}}$. $\square$