Gluing derived equivalences together with bimodules

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Abstract

We fix a commutative ring $\Bbbk$ and a small category I with I_0 (resp. I_1) the class of objects (resp. morphisms). Consider the bicategory $\Bbbk$-$\mathbf{Cat}^{\mathrm{b}}$ of all small $\Bbbk$-categories whose 1-morphisms are the bimodules over them and whose 2-morphisms are the bimodule morphisms (more precisely 1-morphisms from $\mathcal{C}$ to $\mathcal{D}$ are the $\mathcal{D}$-$\mathcal{C}$-bimodules ${}_{\mathcal{D}}M_{\mathcal{C}}$ and the composite $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ is given by the tensor product over $\mathcal{D}$). We define (a generalized version of) the Grothendieck construction $\mathrm{Gr}(X)$ of a lax functor $X\colon I \to \Bbbk$-$\mathbf{Cat}^{\mathrm{b}}$, which enables us to construct new k-categories by tying $\Bbbk$-categories $X(i)$ $(i \in I_0)$ together with bimodules $X(a)$ $(a \in I_1)$. In particular, this construction can present a triangular matrix algebra, or more generally the tensor algebra of a $\Bbbk$-species. For a lax functor $X : I \to \Bbbk$-$\mathbf{Cat}^{\mathrm{b}}$, we define its "module category" $\operatorname{Mod} X$ and its "derived module category" $\mathcal{D}(\operatorname{Mod} X)$, both of which are again lax functors from I. We also define a notion of derived equivalences between lax functors $I \to \Bbbk$-$\mathbf{Cat}^{\mathrm{b}}$. When $\Bbbk$ is a field, we can construct a derived equivalence between the Grothendieck constructions $\mathrm{Gr}(X)$ and $\mathrm{Gr}(X')$ of lax functors $X, X' : I \to \Bbbk$-$\mathbf{Cat}^{\mathrm{b}}$ by gluing derived equivalences between $X(i)$ and $X'(i)$ $(i \in I_0)$ together with bimodules $X(a)$ and $X'(a)$ $(a \in I_1)$ if X and X' are derived equivalent.