

# On Products of Conjugacy Classes and Irreducible Characters in Finite Groups

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## Abstract

Let  $G$  be a finite group. For irreducible complex characters  $\chi$  and  $\varphi$  of  $G$  the irreducible constituents of  $\chi\varphi$  is denoted by  $\eta(\chi\varphi)$ . If  $A$  and  $B$  are two conjugacy classes in  $G$ , then  $AB$  is a union of conjugacy classes in  $G$  and  $\eta(AB)$  denotes the number of distinct conjugacy classes of  $G$  contained in  $AB$ . In this paper we investigate the current research on the impact of their  $\eta$ -functions on the structure of  $G$  as well as some similarity between them.

## 1 Introduction

Let  $G$  be a finite group. For  $a \in G$  let  $Cl(a) = \{g^{-1}ag | g \in G\}$  denote the conjugacy class of  $a$  in  $G$ . A subset  $X$  of  $G$  is called  $G$ -invariant if

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2000 Mathematics Subject classification : Primary 20E45, Secondary 20C15

Key words : Conjugacy classes, irreducible characters, products.

$X^G = \{g^{-1}xg | x \in X, g \in G\} = X$ , and in this case  $X$  is the union of some conjugacy classes of  $G$ . By definition  $\eta(X)$  is the number of distinct conjugacy classes of  $G$  contained in  $X$ . If  $Cl(a)$  and  $Cl(b)$  are two conjugacy classes of  $G$ , then it is clear that  $Cl(a)Cl(b) = \{a^x b^y | x, y \in G\}$  is a  $G$ -invariant subset of  $G$ , hence  $Cl(a)Cl(b)$  contains at least one conjugacy class of  $G$ , i.e.  $\eta(Cl(a)Cl(b)) \geq 1$ . If we know  $\eta(AB)$  for some conjugacy classes  $A$  and  $B$  of  $G$ , then we may ask about the structure of  $G$  as well as the conjugacy classes  $A$  and  $B$ . There are conjectures and unsolved problems concerning the products of conjugacy classes of finite groups. In [5] a conjecture of J. G. Thompson is mentioned that asserts for any nonabelian finite simple group  $G$ , there is a conjugacy class  $C$  for which  $C^2 = G$ , i.e.  $\eta(C^2) = k$  the number of conjugacy classes of  $G$ . It is also conjectured by the authors of [5] that the product of two nontrivial conjugacy classes of a nonabelian simple group is not a single conjugacy class.

Similar concept can be defined for complex characters of a finite group  $G$ . By  $Irr(G)$  we mean the set of all the complex irreducible characters of  $G$ . It is known that for any character  $\chi$  of  $G$  we have  $\chi = \sum_{i=1}^k m_i \chi_i$ , where  $m_i \in \mathbb{N}$  and  $\chi_i \in Irr(G)$ . The irreducible characters  $\chi_i$  are called the constituents of  $\chi$  and similar to the case of product of conjugacy classes we set  $\eta(\chi) = k$ . In particular if  $\chi$  and  $\varphi$  are irreducible characters of  $G$ , then  $\chi\varphi$  is a character of  $G$ , hence  $\eta(\chi\varphi) \geq 1$ . Similar questions as in the case of conjugacy classes may be asked, for instance if we know  $\eta(\chi\varphi)$  for some irreducible characters  $\chi$  and  $\varphi$ , then we may ask about the structure of  $G$  as well as the irreducible characters  $\chi$  and  $\varphi$ . In this case there are conjectures and unsolved problems. For example in [11] it is conjectured that if the product of two faithful irreducible characters of a solvable group is irreducible, then the group is cyclic.

Analog result on the products of the irreducible characters and conjugacy classes of a finite group  $G$  may be true or false, but it is interesting to study

these similarities where either one is considered. In contrast to the case of characters it is proved in [12] that if  $n \geq 5$  is a square number, then there are irreducible characters  $\chi$  and  $\varphi$  of  $\mathbb{A}_n$  such that  $\chi\varphi$  is also an irreducible character of  $\mathbb{A}_n$ .

our aim in this paper is to review the results on the products of irreducible and conjugacy classes of finite groups. We also look at some analog results between the two cases.

## 2 Conjugacy Classes

For a finite group  $G$  it is important to find the  $\eta$ -function on the product of two conjugacy classes of  $G$ . If  $P$  is a Sylow  $p$ -subgroup of  $GL_4(q)$ ,  $q = p^m$ ,  $p$  prime, then  $|P| = q^6 = p^{6m}$  and in [8] it is proved:

**Theorem 1** *If  $A$  and  $B$  are two conjugacy classes of the group  $P$ , then  $\eta(AB) = 1, q, 2q - 1$  or  $q^2 + q - 1$ .*

If  $Q$  is a Sylow  $p$ -subgroup of  $SP_4(q)$ ,  $q = p^m$ ,  $p$  prime, then  $|Q| = q^6 = p^{6m}$  and in [8] we have proved:

**Theorem 2** *If  $A$  and  $B$  are two conjugacy classes of the group  $Q$ , then  $\eta(AB) = 1, q$  or  $2q - 1$  if  $p$  is odd and  $\eta(AB) = 1, q$  or  $q^2$  if  $p = 2$ .*

**Conjecture 1** *The product of two nontrivial conjugacy classes of a non-abelian finite simple group is not a conjugacy class.*

*The conjecture is proved [5] for the following finite simple groups:*

$\mathbb{A}_n$ ,  $n \geq 5$ ,  $Sz(2^{2n+1})$ ,  $n \geq 1$ ,  $PSL_2(q)$ ,  $q = p^n > 2$ ,  $p$  prime, nonabelian simple groups of order less than one million,, all the sporadic simple groups. But for the almost simple groups we have the following result in [4].

**Theorem 3** *Let  $n > 5$ . Given any nontrivial elements  $\alpha$  and  $\beta$  in  $\mathbb{S}_n$ , then  $\eta(\alpha\beta) \geq 2$ , that is to say the product of any two nontrivial conjugacy classes of  $\mathbb{S}_n$  is never a conjugacy class.*

The following results are proved in [3] for  $GL_n(q)$  and  $SL_n(q)$ .

**Theorem 4** *If  $A$  and  $B$  are non-scalar matrices in  $GL_n(q)$ , then  $\eta(Cl(A)Cl(B)) \geq q - 1$ .*

**Theorem 5** *If  $A$  and  $B$  are non-scalar matrices in  $SL_n(q)$ , then  $\eta(Cl(A)Cl(B)) \geq [q/2]$ .*

therefore we see that if  $A$  and  $B$  are non-central elements in  $GL_n(q)$ , than if  $q \neq 2$ ,  $AB$  is not a conjugacy class and if  $A$  and  $B$  are taken as non-central elements in  $GL_n(q)$  and  $q \neq 2, 3$ , then  $AB$  is not a conjugacy class. However in the cases  $q = 2$  and  $3$  the problem is still open for  $SL_n(q)$  and  $GL_n(q)$ .

Relation with derived length appears in papers by Bante and co-workers.

**Theorem 6** *For any finite supersolvable group  $G$  and any conjugacy class  $A$  of  $G$  we have  $dl(\frac{G}{C_G(A)}) \leq 2\eta(AA^{-1}) - 1$ .*

**Theorem 7** *For any finite supersolvable group  $G$  and any conjugacy classes  $A, B$  of  $G$  such that  $AB \cap Z(G) \neq \emptyset$ , we have  $dl(\frac{G}{C_G(A)}) \neq 2\eta(AB) - 1$ .*

Regarding the above theorems the following conjecture is put forward by Bante.

**Conjecture 2** *There exist universal constants  $q$  and  $r$  such that for any finite solvable group  $G$  and any conjugacy classes  $A$  of  $G$ , we have  $dl(\frac{G}{C_G(A)}) \leq q\eta(AA^{-1}) + r$ .*

### 3 Characters

In [6] finite non-abelian groups  $G$  with a faithful irreducible character  $\chi$  such that  $Irr(\chi^2) \subseteq Lin(G)$  is studied and it is asked to do the same for  $Irr(\chi\bar{\chi}) \subseteq Lin(G)$ , where  $\bar{\chi}$  is the conjugate character of  $\chi$ . In [7] a generalization of the above results is proved.

**Theorem 8** *Let  $G$  be a finite group and  $\chi \in Irr(G)$ , and  $m$  and  $n$  be non-negative integers so that  $m + n > 0$ . Then*

- (i)  $Irr(\chi^n\bar{\chi}^m) \subseteq Lin(G)$  if and only if  $G' \leq Z(G)$  and  $[G' : ker\chi]$  divides  $n - m$ .
- (ii) If  $Irr(\chi^n\bar{\chi}^m) \subseteq Lin(G)$ , then all irreducible constituents of  $\chi^n\bar{\chi}^m$  have the same multiplicity.

In [1] and [2] the following results about  $p$ -groups are proved:

**Theorem 9** *Let  $G$  be a finite  $p$ -group, where  $p$  is a prime number. Let  $\chi, \varphi \in Irr(G)$  be faithful characters. then either  $\eta(\chi\varphi) = 1$  or  $\eta(\chi\varphi) \leq (p + 1)/2$ .*

**Theorem 10** *Let  $G$  be a finite  $p$ -group and  $\chi \in Irr(G)$ . Then one of the following holds:*

- (i)  $\chi(1) = 1$  and  $\eta(\chi\bar{\chi}) = 1$ ;
- (ii)  $\chi(1) = p$  and  $\eta(\chi\bar{\chi}) = 2p - 1$  or  $p^2$ ;
- (iii)  $\chi(1) \geq p^2$  and  $\eta(\chi\bar{\chi}) \geq 4p - 3$ ;

Concerning irreducible constituents of the square of an irreducible character the following result is proved in [9].

**Theorem 11** (a) *Let  $G$  be a finite group of odd order and  $\chi \in Irr(G)$ . Then  $\chi$  vanishes on  $G - Z(\chi)$  if and only if  $\eta(\chi^2) = 1$ ;*

- (b) If  $\chi$  and  $\varphi$  are irreducible characters of  $G$  and  $\chi$  vanishes on  $G - Z(\chi)$ , then
- (i) If  $\chi\varphi \cap \text{Lin}(G) \neq \emptyset$ , then  $\eta(\chi\varphi) = |\text{Irr}(G/Z(G))|$ ;
- (ii)  $\chi(1)$  is an odd number, then  $\eta(\chi^2) = 1$ .

The following conjecture is put forward in [!!!!].

**Conjecture 3** Let  $G$  be a finite group of odd order and  $\chi \in \text{Irr}(G)$ , then  $\eta(\chi^2) \leq \frac{1}{2}(\chi(1) + 1)$

## 4 Duality

By duality we mean similar results about products of characters and conjugacy classes of a group  $G$ . With this respect the dual of a conjugacy class  $A$  is an irreducible character  $\chi$  and the dual of  $C_G(A)$  is kernel of the irreducible character  $\chi$ , i.e.  $\text{Ker}(\chi)$ , and the dual of the conjugacy class  $A^{-1}$  is  $\bar{\chi}$ , and the dual of  $Z(G)$  is  $\text{Lin}(G)$ .

The dual of the Theorem 6 is the following:

**Theorem 12** There exist constant  $c$  and  $d$  such that for any finite solvable group  $G$  and any irreducible character  $\chi$  of  $G$ , we have  $dl(\frac{G}{\text{Ker}\chi}) \leq c\eta(\chi\bar{\chi}) + d$ .

The dual of Theorem 8 is the following:

**Theorem 13** A nonabelian group  $G$  possesses a conjugacy class  $C$  with  $|C| \neq 1$  and  $(C^{-1})^m C^n \subseteq Z(G)$  if and only if  $Z(\frac{G}{Z(G)}) \neq 1$  and there exists a non-identity element in the Center of  $\frac{G}{Z(G)}$  whose order divides  $|n - m|$ .

In [10] the following dual results are proved.

**Theorem 14** *Let  $G$  be a finite group. Then  $C^2 \cap Z(G) = \emptyset$  for all conjugacy classes  $C$  with  $|C| \neq 1$  if and only if  $G = O \times T$  where  $O$  is a nonabelian group of odd order and  $T$  is an abelian 2-group.*

**Theorem 15** *Let  $G$  be a finite group. Then  $\text{Irr}(\chi^2) \cap \text{Lin}(G) = \emptyset$  for all  $\chi \in \text{Irr}(G)$  if and only if  $G = O \times T$  where  $O$  is a nonabelian group of odd order and  $T$  is an abelian 2-group.*

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