K-modal BL-logic and Some of it's extensions

Omid Yousefi Kia

Esfahan
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The algebraic view of *BL*-logics has been studied and investigated by some authors [1, 6]. In order to answer the question, "what is an algebraic counterpart of a fuzzy modal logic in Hájek's sense?".
We must firstly construct the algebraic counterpart of fuzzy minimal modal logic *K*, as the minimal modal logic is that of modal logic that satisfies only the axiom K:$\Box(\phi \Rightarrow \psi) \Rightarrow (\Box\phi \Rightarrow \Box\psi)$ among modal axioms. Moreover, every other modal logic can be obtained by extending this system through a (possibly infinite) set of extra axioms [12].

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The fuzzy modal logic $S5(\mathscr{C})$, which was constructed by Hájek, used a schematic extension of *BL*-algebras in order to establish the fuzzy modal logic of *S*5 [15].
The algebraic view of *BL*-logics has been studied and investigated by some authors [1, 6]. In order to answer the question, "what is an algebraic counterpart of a fuzzy modal logic in Hájek's sense?".
We must firstly construct the algebraic counterpart of fuzzy minimal modal logic *K*, as the minimal modal logic is that of modal logic that satisfies only the axiom K:$\Box(\phi \Rightarrow \psi) \Rightarrow (\Box\phi \Rightarrow \Box\psi)$ among modal axioms. Moreover, every other modal logic can be obtained by extending this system through a (possibly infinite) set of extra axioms [12].
The above idea motivated us to introduce an algebraic structure satisfying only the algebraic property of modal principle *K*. Therefore we enrich *BL*-algebras by modal operators to get algebras named *K*-modal *BL*-algebras, which is the algebraic counterpart of fuzzy minimal modal logic [9]. Then we construct a logic which corresponds to *K*-modal *BL*-algebra named *K*-modal *BL*-logic. Furthermore, we will introduce two schematic extensions of *K*-modal *BL*-logic, such as *T*-modal *BL*-logic and *S*4-modal *BL*-logic.

Abtract

In fact, we introduce the fuzzy minimal modal algebra in Hajek's view which it is called K-modal BL-algebra for abbreviation.The properties of this algebra and some types of it's filters are introduced.
Then we obtain the logic corresponding to this algebra.
We introduce some extensions of the K-modal BL-logic such as T-modal BL-logic and $S4$-modal BL-logic. Properties of these logics are verified. We obtain the algebraic semantics of these logics. The algebraic semantics of T-modal BL-logic and $S4$-modal BL-logic is called T-modal BL-algebra and $S4$-modal BL-algebra, respectively. Then we get some properties of these algebras and the relationship between them is obtained.

Definition of K-modal BL-algebra

Definition

Consider a BL-algebra $\mathscr{A} = (A, \cup, \cap, *, \rightarrow, 0, 1)$, we define a unary operator $\Box$ on $\mathscr{A}$, where $\Box : A \rightarrow A$ satisfies the following conditions:

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where $\leqslant$ is defined as $x \leqslant y$ iff $x \cap y = x$, for all $x, y \in A$.

Lemma

Let $\mathscr{M} = (\mathscr{A}, \Box)$ such that the operator $\Box : A \to A$ satisfies the conditions, $(\Box 3)$- $(\Box 1)$ for all $x, y \in A$, then

$$\Box(x \to y) \leqslant \Box x \to \Box y.$$

Remark.
The relation $\Box(x \to y) \leqslant (\Box x \to \Box y)$ is the algebraic properties of the normal principle $K : \Box(\phi \Rightarrow \psi) \Rightarrow (\Box\phi \Rightarrow \Box\psi)$ of modal logics, where ϕ and ψ are formulas of the related language.
Since the algebra $\mathscr{M} = (\mathscr{A}, \Box)$ satisfies the algebraic counterpart of principle K, we used the sign K for the name of the algebra $\mathscr{M}$.

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Since the algebra $\mathscr{M} = (\mathscr{A}, \Box)$ satisfies the algebraic counterpart of principle K, we used the sign K for the name of the algebra $\mathscr{M}$.
The algebra $\mathscr{M} = (\mathscr{A}, \Box)$, is called a K-modal BL-algebra provided that $\Box$ satisfies the conditions ($\Box$1)-($\Box$3).
From now on, we denote the K-modal BL-algebra by $\mathscr{M} = (\mathscr{A}, \Box)$.

Example

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Consider $\mathscr{A} = (\{0, a, b, c, 1\}, \cap, \cup, *, \rightarrow, 0, 1)$ with lattice oreder $0 < a < b < 1$ and $a < c < 1$. This structure together with the following operations is a BL-algebra:

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

$*$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	a	b
c	0	a	a	c	c
1	0	a	b	c	1

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0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	a	b
c	0	a	a	c	c
1	0	a	b	c	1

We define the unary operation $\Box$ on $\mathscr{A}$ as:

x	0	a	b	c	1
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1	0	a	b	c	1

We define the unary operation $\Box$ on $\mathscr{A}$ as:

x	0	a	b	c	1
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Then the structure $(\mathscr{A}, \Box)$ is a K-modal BL-algebra.

Example 1.2.

Example

Define on the real unit interval $I = [0, 1]$ the binary operations $*$ and $\to$ as follows:

$$x * y = \max(0, x + y - 1)$$
$$x \to y = \min(1, 1 - x + y)$$

Then $(I, \cap, \cup, *, \to, 0, 1)$ is a BL-algebra (called Lukasiewicz structure)

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Now, we define an operator $\Box$ on this structure as follow:

$$\Box x = \begin{cases} 1 & \text{if} \quad x = 1 \\ \frac{1}{2}x & \text{if} \quad x \neq 1 \end{cases}$$

Let $x, y \neq 1$ then we get $\Box x * \Box y = \frac{1}{2}x * \frac{1}{2}y = \max(0, \frac{1}{2}x + \frac{1}{2}y - 1) = 0 \leqslant \frac{1}{2}\max(0, x + y - 1) = \frac{1}{2}(x * y) = \Box(x * y)$. This shows that the $\Box 1$ holds. If $x = 1$ or $y = 1$ then clearly the axiom $\Box 1$ holds.
We can easily verify that the axioms $\Box 2$ and $\Box 3$ hold.

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We can easily verify that the axioms $\Box 2$ and $\Box 3$ hold.
Then the structure $(I, \leqslant, *, \to, 0, 1, \Box)$ is a K-modal BL-algebra.

Remark

1. If $\Box 4$: $\Box(x * y) = \Box x * \Box y$, then $\Box 4$ implies $\Box 1$ and $\Box 2$. But $\Box 1$ and $\Box 2$ do not imply $\Box 4$ generally. Indeed, if $\Box 4$ holds then clearly $\Box 4$ implies $\Box 1$. If in the previous Examplewe take $x = \frac{1}{2}$ and $y = \frac{3}{4}$ then $\Box x * \Box y \neq \Box(x * y)$, but $\Box 1$ and $\Box 2$ hold.

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1. If $\mathscr{A} = (A, \cap, \cup, *, \rightarrow, 0, 1)$ is a *BL*-algebra and $\mathscr{B}(A)$ is the set of all complemented elements of *BL*-algebra $\mathscr{A}$ then $e * x = e \cap x$ for each $e \in \mathscr{B}(A)$ and $x \in A$.
Hence the condition $\Box 4 : \Box(x * y) = \Box x * \Box y$ reduces to the condition $(1) : \Box(x \cap y) = \Box x \cap \Box y$ of the Definition of modal algebra.

Example

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Consider the structure $\mathscr{A}$ of example 1.2.
Case1. Define the unary operation $\Box$ on $\mathscr{A}$ as:

x	0	a	b	c	1
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Then the structure $(\{0, a, b, c, 1\}, \cap, \cup, *, \rightarrow, 0, 1, \Box)$, i.e. $(\mathscr{A}, \Box)$ is not a K-modal BL-algebra.

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Then the structure $(\{0, a, b, c, 1\}, \cap, \cup, *, \to, 0, 1, \Box)$, i.e. $(\mathscr{A}, \Box)$ is not a K-modal BL-algebra.
We can easily check that $\Box$2 and $\Box$3 are verified, but $\Box$1 does not hold. In fact if $x = b$ and $y = c$, we have $x * y = b * c = a$, $\Box(x * y) = \Box a = 0$, $\Box x * \Box y = \Box b * \Box c = a * a = a$ and $a \not\leq 0$. This shows that the axiom $\Box$1 is independent of the other axioms.

Example

Case2. Define the unary operator $\Box$ on $\mathscr{A}$ as:

x	0	a	b	c	1
$\Box$	0	0	0	0	0

The axioms BL, $\Box 1$, $\Box 2$ hold, but the axiom $\Box 3$ does not hold, i.e., this case shows that the axiom $\Box 3$ is independent of the other axioms.

Example

Case3. If the unary operator $\Box$ on $\mathscr{A}$ is defined as:

x	0	a	b	c	1
$\Box$	0	b	a	b	1

Then the axioms BL, $\Box 1$, $\Box 3$ hold, but the axiom $\Box 2$ does not hold for $x = a$ and $y = b$. This case shows that the axiom $\Box 2$ is independent of the other axioms.

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The following identity is true in each K-modal BL-algebra.

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Theorem

The class of all K-modal BL-algebras is a variety of algebras.

Some properties of K-modal BL-algebras

In each K-modal BL-algebra the following properties hold:

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(10) $\Box((y \to x) \to z) \leqslant \Box((x \to y) \to z) \to \Box z$.

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The deduction rule of *BL* is modus ponens. Given this, the notions of a proof and provable formula in *BL* are defined in the obvious way. Needless to say the connectives are $\Rightarrow$ and &. Further connectives are defined as follows:

$$\phi \wedge \psi \text{ is } \phi \& (\phi \Rightarrow \psi);$$

$$\phi \vee \psi \text{ is } ((\phi \Rightarrow \psi) \Rightarrow \psi) \wedge ((\psi \Rightarrow \phi) \Rightarrow \phi);$$

$$\neg\phi \text{ is } \phi \Rightarrow \bar{0};$$

$$\phi \equiv \psi \text{ is } (\phi \Rightarrow \psi) \& (\psi \Rightarrow \phi).$$

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*MBL*2) $(\phi \Rightarrow \psi) \Rightarrow (\Box\phi \Rightarrow \Box\psi)$;

Language of *K*-modal *BL*-logic

The language of the *K*-modal *BL*-logic (*KMBL*-logic, for short), $\mathscr{L}$, is the language of *BL*-logic expanded by the unary connective $\Box$. Axioms of *K*-modal *BL*-logic are those of *BL*-logic plus

*MBL*1) $\Box\phi \& \Box\psi \Rightarrow \Box(\phi\&\psi)$;

*MBL*2) $(\phi \Rightarrow \psi) \Rightarrow (\Box\phi \Rightarrow \Box\psi)$;

*MBL*3) $\Box\bar{1}$.

Deduction rules of *K*-modal *BL*-logic are modus ponens and necessitation, i.e., from ϕ we derive $\Box\phi$.
Let $F_{\mathscr{L}}$ be the set of all formulas in the language $\mathscr{L}$ and let $\mathscr{M} = (\mathscr{A}, \Box)$.
A truth evaluation of formulas is a mapping $e : F_{\mathscr{L}} \to A$, defined as follows:
If ϕ is a propositional variable p then $e(p) \in A$.

This extends in the obvious way to an evaluation of all formulas using the operations on $\mathscr{M}$ as truth functions, i.e.,

$$
\begin{aligned}
&e(\bar{0}) = 0,\\
&e(\bar{1}) = 1,\\
&e(\phi \Rightarrow \psi) = e(\phi) \to e(\psi),\\
&e(\phi \& \psi) = e(\phi) * e(\psi),\\
&e(\phi \wedge \psi) = e(\phi) \cap e(\psi),\\
&e(\phi \vee \psi) = e(\phi) \cup e(\psi),\\
&e(\neg \phi) = e(\phi) \to 0,\\
&e(\Box \phi) = \Box e(\phi)
\end{aligned}
$$

for all formulas $\phi, \psi \in F_{\mathscr{L}}$.

K-modal BL-logic, satisfies normal principle

Theorem

The (modal) principle

$$K : \Box(\phi \Rightarrow \psi) \Rightarrow (\Box\phi \Rightarrow \Box\psi)$$

is provable in the K-modal BL-logic.

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Since the connectives & and $\Rightarrow$ are used in the two axioms but in the axiom K the connective $\Rightarrow$ is only used.

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Since the connectives & and $\Rightarrow$ are used in the two axioms but in the axiom K the connective $\Rightarrow$ is only used.
Needless to say that the existence of axiom $(KMBL3)$ in previous Definition is necessary, because necessity of any tautology is a tautology.

The classes of provably equivalent of formulas

Now, we show that the classes of provably equivalent formulas form a K-modal BL-algebra.

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Let T be a theory over *K*-modal *BL*-logic. For each formula ϕ, let $[\phi]_T$ be the set of all formulas ψ such that $T \vdash \phi \equiv \psi$ and M_T be the set of all the classes $[\phi]_T$.
We define:
$0 = [\bar{0}]_T$, $1 = [\bar{1}]_T$,
$[\phi]_T * [\psi]_T = [\phi \& \psi]_T$,
$[\phi]_T \to [\psi]_T = [\phi \Rightarrow \psi]_T$,
$[\phi]_T \cap [\psi]_T = [\phi \wedge \psi]_T$,
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$\Box[\phi]_T = [\Box\phi]_T$.
This algebra is denoted by $\mathbf{M}_T$.

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Lemma 2.8. $\mathbf{M}_T$ *is a K-modal BL-algebra.*

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$[\phi]_T \cap [\psi]_T = [\phi \wedge \psi]_T$,
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This algebra is denoted by $\mathbf{M}_T$.

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Lemma 2.8. $\mathbf{M}_T$ *is a K-modal BL-algebra.*

Lemma 2.9. All axioms of *KMBL*-logic are $\mathscr{M}$-tautology, for every *K*-modal *BL*-algebra $\mathscr{M}$.

Soundness and Completeness

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Let $e : F_{\mathscr{L}} \to A$ be a truth evaluation:

(1) If $e(\phi) = 1$ and $e(\phi \Rightarrow \psi) = 1$ then $e(\psi) = 1$;

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(2) If $e(\phi) = 1$ then $e(\Box\phi) = 1$, for any formula ϕ and ψ.

(Completeness).
Theorem 2.11. The *K*-modal *BL*-logic is complete, i.e., the following are equivalent for every formula ϕ:

(1) $KMBL \vdash \phi$;
(2) for each *K*-modal *BL*-algebra $\mathscr{M}$, ϕ is an $\mathscr{M}$-tautology.

T-modal *BL*-logic

In this section we introduce the *T*-modal *BL*-logic (*TMBL*-logic, for short). In fact the *T*-modal *BL*-logic is an extension of the *K*-modal *BL*-logic by adding two extra axioms to the axioms of *K*-modal *BL*-logic as follows:

$$\begin{array}{ll} (\mathit{TMBL}1) & \Box(\phi \& \psi) \Rightarrow \Box\phi \& \Box\psi; \\ (\mathit{TMBL}2) & \Box\phi \Rightarrow \phi. \end{array}$$

The language of *TMBL*-logic is the same language of *KMBL*-logic and the truth evaluation *e* and the set of formulas $F_{\mathscr{L}}$ are defined in the same way. Deduction rules are modus ponens and necessitation.

Definition 3.2. A T-modal BL-algebra, ($TMBL$-algebra, for short) is a $KMBL$-algebra $\mathscr{M} = (\mathscr{A}, \Box)$, in which the following formulas are true:

Algebraic semantics of *T*-modal *BL*-logics

Definition 3.2. A *T*-modal *BL*-algebra, (*TMBL*-algebra, for short) is a *KMBL*-algebra $\mathscr{M} = (\mathscr{A}, \Box)$, in which the following formulas are true:

$$(\Box 4)\ \Box(x * y) = \Box x * \Box y;$$
$$(\Box 5)\ \Box x \leqslant x.$$

Algebraic semantics of T-modal BL-logics

Definition 3.2. A T-modal BL-algebra, ($TMBL$-algebra, for short) is a $KMBL$-algebra $\mathscr{M} = (\mathscr{A}, \Box)$, in which the following formulas are true:

$$(\Box 4)\ \Box(x * y) = \Box x * \Box y;$$
$$(\Box 5)\ \Box x \leqslant x.$$

Clearly, every $TMBL$-algebra is a $KMBL$-algebra but the converse is not true generally.

Example

Example

Consider $\mathscr{A} = (\{-1, 0, a, b, c, d, 1\}, \cap, \cup, *, \rightarrow, 0, 1)$. This structure together with the following operations is a *BL*-algebra

$\rightarrow$	-1	0	a	b	c	d	1
-1	1	1	1	1	1	1	1
0	-1	1	1	1	1	1	1
a	-1	d	1	d	1	d	1
b	-1	c	c	1	1	1	1
c	-1	b	c	d	1	d	1
d	-1	a	a	c	c	1	1
1	-1	0	a	b	c	d	1

Example

$*$	-1	0	a	b	c	d	1
-1	-1	-1	-1	-1	-1	-1	-1
0	-1	0	0	0	0	0	0
a	-1	0	a	0	a	0	a
b	-1	0	0	0	0	b	b
c	-1	0	a	0	a	b	c
d	-1	0	0	b	b	d	d
1	-1	0	a	b	c	d	1

Example

$*$	-1	0	a	b	c	d	1
-1	-1	-1	-1	-1	-1	-1	-1
0	-1	0	0	0	0	0	0
a	-1	0	a	0	a	0	a
b	-1	0	0	0	0	b	b
c	-1	0	a	0	a	b	c
d	-1	0	0	b	b	d	d
1	-1	0	a	b	c	d	1

We define the unary operation $\Box$ on $\mathscr{A}$ as:

x	-1	0	a	b	c	d	1
$\Box$	-1	-1	-1	-1	c	-1	1

We can easily verify that the K-modal BL-algebra $\mathscr{M} = (\mathscr{A}, \Box)$ is a T-modal BL-algebra.

Example

Example

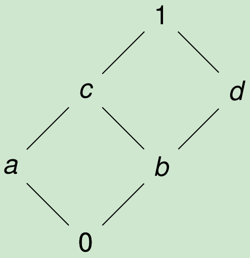
Consider $\mathscr{A} = (\{0, a, b, c, d, 1\}, \cap, \cup, *, \rightarrow, 0, 1)$. This structure together with the following operations is a *BL*-algebra:

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

Example

Hasse diagram of BL-algebra $\mathscr{A}$ is as:



Example

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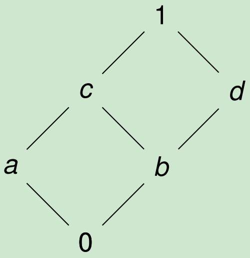
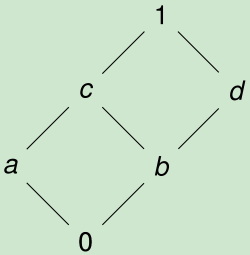


We define the unary operation $\square$ on $\mathscr{A}$ as:

x	0	a	b	c	d	1
$\square$	0	1	0	1	0	1

Example

Hasse diagram of BL-algebra $\mathscr{A}$ is as:



We define the unary operation $\Box$ on $\mathscr{A}$ as:

x	0	a	b	c	d	1
$\Box$	0	1	0	1	0	1

We can easily verify that $\mathscr{M} = (\mathscr{A}, \Box)$ is a K-modal BL-algebra which satisfies all of the conditions ($\Box$1)-($\Box$4) but the condition ($\Box$5) does not hold. Hence the K-modal BL-algebra $\mathscr{M} = (\mathscr{A}, \Box)$ is not T-modal BL-algebra. Moreover, this example shows that the condition ($\Box$5) is independent of other conditions.

Example 3.5.

Example

Consider $\mathscr{A} = (\{-2, -1, 0, a, b, c, d, 1\}, \cap, \cup, *, \rightarrow, 0, 1)$. This structure together with the following operations is a *BL*- algebra:

$\rightarrow$	-2	-1	0	*a*	*b*	*c*	*d*	1
-2	1	1	1	1	1	1	1	1
-1	-1	1	1	1	1	1	1	1
0	-2	-1	1	1	1	1	1	1
a	-2	-1	*d*	1	*d*	1	*d*	1
b	-2	-1	*a*	*a*	1	1	1	1
c	-2	-1	0	*a*	*d*	1	*d*	1
d	-2	-1	*a*	*a*	*c*	*c*	1	1
1	-2	-1	0	*a*	*b*	*c*	*d*	1

Example

$*$	-2	-1	0	a	b	c	d	1
-2	-2	-2	-2	-2	-2	-2	-2	-2
-1	-2	-2	-1	-1	-1	-1	-1	-1
0	-2	-1	0	0	0	0	0	0
a	-2	-1	0	a	0	a	0	a
b	-2	-1	0	0	b	b	b	b
c	-2	-1	0	a	b	c	b	c
d	-2	-1	0	0	b	b	d	d
1	-2	-1	0	a	b	c	d	1

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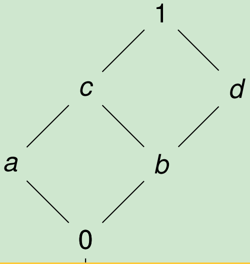

Example

We define the unary operation $\Box$ on $\mathscr{A}$ as:

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x	-2	-1	0	a	b	c	d	1
$\Box$	-2	-1	0	a	-1	a	d	1

Example

We define the unary operation $\Box$ on $\mathscr{A}$ as:

x	-2	-1	0	a	b	c	d	1
$\Box$	-2	-1	0	a	-1	a	d	1

We can easily verify that $\mathscr{M} = (\mathscr{A}, \Box)$ is a K-modal BL-algebra which satisfies all of the conditions ($\Box$1)-($\Box$5) except ($\Box$4), since
$\Box(c * d) = \Box b = -1 \neq 0 = \Box c * \Box d$.

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We define the unary operation $\Box$ on $\mathscr{A}$ as:

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$\Box(c * d) = \Box b = -1 \neq 0 = \Box c * \Box d$.
Moreover, this example shows that the condition ($\Box$4) is independent of other conditions.

(Completeness). *TMBL*-logic is complete, i.e., For every formula $\phi \in F_{\mathscr{L}}$, the following are equivalent:

(1) $TMBL \vdash \phi$;

(Completeness). *TMBL*-logic is complete, i.e., For every formula $\phi \in F_{\mathscr{L}}$, the following are equivalent:

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(2) for each *T*-modal *BL*-algebra $\mathscr{M}$, ϕ is an $\mathscr{M}$-tautology.

$S4$-modal BL-logic

In this section we introduce the $S4$-modal BL-logic ($S4MBL$-logic, for short). In fact, the $S4$-modal BL-logic is an extension of the K-modal BL-logic by adding five extra axioms to the axioms of K-modal BL-logic as follows:

$$
\begin{array}{ll}
(TMBL1) & \Box(\phi \& \psi) \Rightarrow \Box\phi \& \Box\psi; \\
(TMBL2) & \Box\phi \Rightarrow \phi; \\
(S4MBL3) & \Box\phi \Rightarrow \Box\Box\phi; \\
(S4MBL4) & \Box(\phi \vee \psi) \Rightarrow \Box\phi \vee \Box\psi; \\
(S4MBL5) & (\Box\phi \vee \Box\psi) \Rightarrow \Box(\phi \vee \psi).
\end{array}
$$

The language of $S4MBL$-logic is the same language of $KMBL$-logic and the truth evaluation e and the set of formulas $F_{\mathscr{L}}$ are defined in the same way. Deduction rules are modus ponens and necessitation.

Algebraic semantics of $S4$-modal BL-logics

A $S4$-modal BL-algebra, ($S4MBL$-algebra, for short) is a $KMBL$-algebra $\mathscr{M} = (\mathscr{A}, \Box)$, in which the following formulas are true:

$$(\Box 4)\ \Box(x * y) = \Box x * \Box y;$$

$$(\Box 5)\ \Box x \leqslant x;$$

$$(\Box 6)\ \Box x \leqslant \Box\Box x;$$

$$(\Box 7)\ \Box(x \cup y) = \Box x \cup \Box y.$$

Algebraic semantics of $S4$-modal BL-logics

A $S4$-modal BL-algebra, ($S4MBL$-algebra, for short) is a $KMBL$-algebra $\mathscr{M} = (\mathscr{A}, \Box)$, in which the following formulas are true:

$$(\Box 4)\ \Box(x * y) = \Box x * \Box y;$$
$$(\Box 5)\ \Box x \leqslant x;$$
$$(\Box 6)\ \Box x \leqslant \Box\Box x;$$
$$(\Box 7)\ \Box(x \cup y) = \Box x \cup \Box y.$$

Example

Consider the unit interval $I = [0, 1]$. We define binary operations $*$, $\to$ and unary operator $\Box$ on I as follow: $x * y = x \cap y$,

$$x \to y = \begin{cases} 1, & x \leqslant y \\ y, & \text{otherwise.} \end{cases}$$

Example

and

$$\Box x = \begin{cases} 0, & 0 \leqslant x < \frac{1}{3} \\ \frac{1}{3}, & \frac{1}{3} \leqslant x < \frac{1}{2} \\ \frac{1}{2}, & \frac{1}{2} \leqslant x < 1 \\ 1, & x = 1. \end{cases}$$

We can easily verify that $\mathscr{I} = (I, \Box)$ is a K-modal BL-algebra which satisfies the conditions ($\Box$4)-($\Box$7). Hence $\mathscr{I} = (I, \Box)$ is a $S4$-modal BL-algebra. Every $S4MBL$-algebra is a $TMBL$-algebra but the converse is not true generally.

Example

Consider $\mathscr{A} = (\{-1, 0, a, b, c, d, 1\}, \cap, \cup, *, \rightarrow, 0, 1)$. This structure together with the following operations is a *BL*-algebra:

$\rightarrow$	-1	0	a	b	c	d	1
-1	1	1	1	1	1	1	1
0	-1	1	1	1	1	1	1
a	-1	d	1	d	1	d	1
b	-1	c	c	1	1	1	1
c	-1	b	c	d	1	d	1
d	-1	a	a	c	c	1	1
1	-1	0	a	b	c	d	1

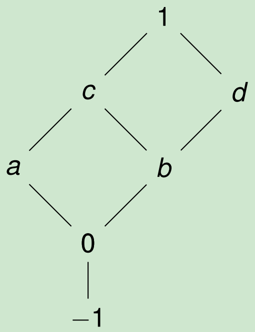
Example

$*$	-1	0	a	b	c	d	1
-1	-1	-1	-1	-1	-1	-1	-1
0	-1	0	0	0	0	0	0
a	-1	0	a	0	a	0	a
b	-1	0	0	0	0	b	b
c	-1	0	a	0	a	b	c
d	-1	0	0	b	b	d	d
1	-1	0	a	b	c	d	1

Example 4.6.

Example

Hasse diagram of the *BL*-algebra $\mathscr{A}$ is as :



We define the unary operation $\Box$ on $\mathscr{A}$ as:

x	-1	0	a	b	c	d	1
$\Box$	-1	-1	-1	-1	c	-1	1

Example

We can easily verify that $\mathscr{M} = (\mathscr{A}, \Box)$ is a *K*-modal *BL*-algebra which satisfies all of the conditions ($\Box$4)-($\Box$6) but the condition ($\Box$7) does not hold. Since $\Box(a \cup b) = \Box c = c \neq -1 = \Box a \cup \Box b$.

Example

We can easily verify that $\mathscr{M} = (\mathscr{A}, \Box)$ is a K-modal BL-algebra which satisfies all of the conditions ($\Box$4)-($\Box$6) but the condition ($\Box$7) does not hold. Since $\Box(a \cup b) = \Box c = c \neq -1 = \Box a \cup \Box b$.
Therefore the K-modal BL-algebra $\mathscr{M} = (\mathscr{A}, \Box)$ is a T-modal BL-algebra, but it is not $S4$-modal BL-algebra.

Example

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Therefore the K-modal BL-algebra $\mathscr{M} = (\mathscr{A}, \Box)$ is a T-modal BL-algebra, but it is not $S4$-modal BL-algebra.
Moreover, this example shows that ($\Box$7) is independent of other conditions.

Example

We can easily verify that $\mathscr{M} = (\mathscr{A}, \Box)$ is a *K*-modal *BL*-algebra which satisfies all of the conditions ($\Box$4)-($\Box$6) but the condition ($\Box$7) does not hold. Since $\Box(a \cup b) = \Box c = c \neq -1 = \Box a \cup \Box b$.
Therefore the *K*-modal *BL*-algebra $\mathscr{M} = (\mathscr{A}, \Box)$ is a *T*-modal *BL*-algebra, but it is not *S*4-modal *BL*-algebra.
Moreover, this example shows that ($\Box$7) is independent of other conditions.

Example

Example 3.6. In the *BL*-algebra
$\mathscr{A} = (\{-2, -1, 0, a, b, c, d, 1\}, \cap, \cup, *, \rightarrow, 0, 1)$ of Example 3.5.
we define the unary operation $\Box$ on $\mathscr{A}$ as:

x	-2	-1	0	a	b	c	d	1
$\Box$	-2	-2	-1	a	0	a	d	1

We can easily verify that $\mathscr{M} = (\mathscr{A}, \Box)$ is a *K*-modal *BL*-algebra which satisfies all of the conditions ($\Box$4)-($\Box$7) except ($\Box$6), since $\Box 0 = -1 \not\leq -2 = \Box\Box 0$.
Moreover, this example shows that ($\Box$6) is independent of other conditions.

Lemma

Lemma 4.8. The algebra $\mathbf{M}_T$ *is a S4MBL-algebra.*

Lemma

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Lemma

Lemma 4.9. All axioms of S4MBL-logic are $\mathscr{M}$*-tautologies, for every S4-modal BL-algebra* $\mathscr{M}$*.*

Lemma

Lemma 4.8. The algebra $\mathbf{M}_T$ is a $S4MBL$-algebra.

Lemma

Lemma 4.9. All axioms of $S4MBL$-logic are $\mathscr{M}$-tautologies, for every $S4$-modal BL-algebra $\mathscr{M}$.

Theorem

(Completeness). $S4MBL$-logic is complete, i.e., For every formula $\phi \in F_{\mathscr{L}}$, the following are equivalent:

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Lemma 4.8. The algebra $\mathbf{M}_T$ is a S4MBL-algebra.

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Lemma 4.9. All axioms of S4MBL-logic are $\mathscr{M}$-tautologies, for every S4-modal BL-algebra $\mathscr{M}$.

Theorem

(Completeness). S4MBL-logic is complete, i.e., For every formula $\phi \in F_{\mathscr{L}}$, the following are equivalent:

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(2) *for each linearly ordered S4-modal BL-algebra $\mathscr{M}$, ϕ is an $\mathscr{M}$–tautology;*

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Theorem 1.10.

Theorem

Theorem1.10. Suppose that $\mathscr{M} = (\mathscr{A}, \Box)$ *be a* K*-modal BL-algebra and* F *be a filter on* $\mathscr{M}$ *such that* $1 \neq a \notin F$ *. Then there exists a* K*-modal prim filter* F' *on* $\mathscr{M}$ *containing* F *and* $a \notin F'$*, provided that* $\Box$ *satisfies four extra conditions:*

$$\Box 4 : \Box x * \Box y = \Box(x * y);$$

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$$\Box 4: \ \Box x * \Box y = \Box(x * y);$$

$$\Box 5: \ \Box x \leqslant x;$$

$$\Box 6: \ \Box x \leqslant \Box\Box x;$$

$$\Box 7: \ \Box(x \cup y) = \Box x \cup \Box y.$$

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$\Box 4$: $\Box x * \Box y = \Box(x * y)$;

$\Box 5$: $\Box x \leqslant x$;

$\Box 6$: $\Box x \leqslant \Box\Box x$;

$\Box 7$: $\Box(x \cup y) = \Box x \cup \Box y$.

Corollary

Corollary1.11. Let $\mathscr{A}$ be a BL-algebra with unary operator $\Box$ satisfying $\Box 3$-$\Box 7$. The construction $\mathscr{M} = (\mathscr{A}, \Box)$ as a special K-modal BL-algebra is a subdirect product of linearly ordered K-modal BL-algebras.

Corollary

Corollary 4.3. Each $S4$-modal BL-algebra is a sub-direct product of a system of linearly ordered $S4$-modal BL-algebras.

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Thanks for your attention.