Thematic Program on

Dynamical Systems

School of Mathematics, IPM, Tehran

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IPM
Institute for Research in Fundamental Sciences

SHORT-COURSE

Ergodic Aspects of Partially Hyperbolic Diffeomorphisms

LORENZO J. DÍAZ & KATRIN GELFERT

PUC-Rio, Brazil   UFRJ, Rio de Janeiro, Brazil

SCHEDULE:	Lecture 1:	Wednesday, Feb. 22, 2017, 15:30–17:00 (Lecture Hall 1)
	Lecture 2:	Thursday, Feb. 23, 2017, 10:30–12:00
	Lecture 3:	Thursday, Feb. 23, 2017, 14:00–15:30
	Lecture 4:	Tuesday, March 7, 2017, 9:30–10:45
	Lecture 5:	Tuesday, March 7, 2017, 14:00–15:30
	Lecture 6:	Wednesday, March 8, 2017, 15:30–17:00 (Lecture Hall 1)
VENUE:	Lecture Hall 2, IPM Niavaran Bldg., Niavaran Square, Tehran	

ABSTRACT. Even though the theory of dynamical systems developed very rapidly during the last decades, at the present state of the art in many aspects one completely understands only low-dimensional (conformal) and uniformly hyperbolic dynamical systems such as, for example, hyperbolic surface diffeomorphisms. In particular, the thermodynamic formalism initiated in the 70s by Sinai, Ruelle, and Bowen, nowadays provides a powerful tool to describe the interplay of topological dynamics with the ergodic theory of measures which are invariant under the dynamics. The principle ingredients of this formalism are entropy, pressure, free energy, equilibrium states, Lyapunov exponents, and Birkhoff averages. It is also intimately related with the theory of fractal dimensions.

The aim of this mini-course is to study higher-dimensional and nonhyperbolic dynamical systems. One expects that the non-hyperbolicity of the systems must be reflected in the existence of non-hyperbolic invariant ergodic measures (i.e., having some zero Lyapunov exponents). We study this question in the setting of the so-called robustly non-hyperbolic transitive diffeomorphisms. We will present first a method based on periodic approximations introduced in [8] in the setting of skew-products, and later used to construct open sets of diffeomorphisms with such measures [9] and applied to generic non-hyperbolic homoclinic classes of diffeomorphisms [7, 2]. This approach provides conditions for a sequence of atomic measures to converge to a nontrivial nonhyperbolic ergodic measure. More recently, [1] introduced a method based on the so-called blenders and flip-flop for constructing nonhyperbolic ergodic measures with positive entropy. We will review these constructions and explain their

Geometry & Topology

main ingredients. Let us emphasise that in the setting of thermodynamical formalism of nonhyperbolic systems nonhyperbolic ergodic measures with positive entropy play an essential role that can not be discarded.

To develop this formalism and in order to understand higher-dimensional and nonhyperbolic dynamical systems, one natural approach is to consider classes of systems with gradually increasing type of complexity. Hence, it is very natural to investigate diffeomorphisms which have a (step) skew-product structure with a low-dimensional surface diffeomorphism as base map and one-dimensional fiber maps.

Motivated by this, we study a family of partially hyperbolic diffeomorphisms of step skew-product type modeled over a horseshoe with interval or circle fiber maps. These systems are genuinely nonhyperbolic containing intermingled horseshoes (and periodic points) of different hyperbolic behavior (contracting and expanding center). The associated invariant set possesses a very rich topological fiber structure, it contains uncountably many trivial and uncountably many nontrivial fibers (see [3]).

Starting with a class of very simple paradigmatic models we study transitive step skew-products modelled over a shift with one-dimensional fiber maps. In case the fiber maps have both contracting and expanding regions, this dynamics is genuinely nonhyperbolic and simultaneously exhibits ergodic measures with positive, negative, and zero exponents. Moreover, it contains a gap (at least one) which is associated with (in fact caused by) an “exposed” ergodic measure, [4, 3].

We introduce a set of axioms to capture the essential dynamic features and analyze its topological dynamics. It turns out that those axioms equivalently characterize nonhyperbolic robustly transitive maps. We also study ergodic and thermodynamic properties by analyzing the space of ergodic hyperbolic (with either expanding or contracting fiber exponent) and nonhyperbolic measures in the weak$*$ topology and in entropy. Our methods include the explicit construction of hyperbolic sets based on an approximation using so-called skeletons, multi-variable-time horseshoes, and our set of axioms, [6, 5].

References

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[6] L. J. Díaz, K. Gelfert, M. Rams, *Nonhyerbolic step skew-products: Entropy spectrum of Lyapunov exponents,* Preprint `arXiv:1610.07167`.

[7] L. J. Díaz, A. Gorodetski, *Non-hyperbolic ergodic measures for non-hyperbolic homoclinic classes* Ergodic Theory and Dynamical Systems, **29** (2009), 1479–1513.

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[9] V. Kleptsyn, M. Nalsky, *Stability of the existence of nonhyperbolic measures for C^1-diffeomorphisms,* Funct. Anal. Appl. **41** (2007), 271–283.