Rotation theory II

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Homeomorphisms of the torus

$$\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2, \quad \pi\colon \mathbb{R}^2 \to \mathbb{T}^2 \text{ projection;}$$
$$f \in \mathrm{Homeo}_+(\mathbb{T}^2), \quad \widetilde{f}\colon \mathbb{R}^2 \to \mathbb{R}^2 \text{ a lift of } f;$$

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Linear and periodic part

- $v \in \mathbb{Z}^2 \implies \widetilde{f}(z+v) - \widetilde{f}(z) := \widetilde{A}(v) \in \mathbb{Z}^2$.
- $v \mapsto \widetilde{A}(v)$ is an automorphism of $\mathbb{Z}^2$, so $\widetilde{A} \in \mathrm{GL}(2,\mathbb{Z})$
- f preserves orientation $\implies \widetilde{A} \in \mathrm{SL}(2,\mathbb{Z})$
- $\widetilde{f}(z) = \widetilde{A}(z) + \widetilde{\Delta}(z)$ where $\widetilde{\Delta}$ is $\mathbb{Z}^2$-periodic.
- $\widetilde{A}$ induces a linear toral automorphism $A\colon \mathbb{T}^2 \to \mathbb{T}^2$
- $\widetilde{\Delta}$ induces $\Delta\colon \mathbb{T}^2 \to \mathbb{R}^2$ (bounded).
- f is isotopic to A.

$\mathbb{Z}^2 \simeq \pi_1(\mathbb{T}^2)$, $A =$ map induced by f on the fundamental group.

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Classification of the "isotopy class" $A \in \mathrm{SL}(2,\mathbb{Z})$:

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- Parabolic: similar to identity case, with some rigidity;

We focus on the case $A = \mathrm{Id}$ (i.e. f isotopic to Id), so $\widetilde{f}(z) = z + \widetilde{\Delta}(z)$.

Homeomorphisms of the torus

- $\widetilde{f}$ commutes with integer translations: $\widetilde{f}(z+v) = \widetilde{f}(z) + v \ \forall v \in \mathbb{Z}^2$;
- Displacement function $\widetilde{\Delta} = \widetilde{f} - \mathrm{Id}$ ($\mathbb{Z}^2$-periodic);
- Induces $\Delta\colon \mathbb{T}^2 \to \mathbb{R}^2$, $\Delta(z) = \widetilde{\Delta}(\tilde{z})$, for $\tilde{z} \in \pi^{-1}(z)$;
- $\Delta^n(z) = \sum_{k=0}^{n-1} \Delta \circ f^k(z) = \widetilde{f}^n(\tilde{z}) - \tilde{z}$ (displacement cocycle).

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Two directions!

Rotation <u>vectors</u> (two-dimensional). Analogous definitions as before.

Rotation sets of toral homeomorphisms

- $z \in \mathbb{T}^2$, $\tilde{z} \in \pi^{-1}(z)$. **Rotation vector of** z (if the limit exists):

$$\rho(\tilde{f}, z) = \lim_{n\to\infty} \frac{\tilde{f}^n(\tilde{z}) - \tilde{z}}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Delta(f^n(z)) = \lim_{n\to\infty} \frac{1}{n} \Delta^n(z)$$

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- Birkhoff: if $\mu \in \mathcal{M}(f)$, then $\rho(\tilde{f}, z)$ exists for μ-a.e. z and

$$\int \rho(\tilde{f}, z)\, d\mu(z) = \int \Delta\, d\mu := \rho(\tilde{f}, \mu) \text{ (\textbf{mean rotation vector for} } \mu\text{)}$$

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- $\rho_m(\tilde{f}) \supset \rho_p(\tilde{f}) \supset \rho_{\mathrm{erg}}(\tilde{f}) := \{\rho(\tilde{f}, \mu) : \mu \text{ ergodic}\}$..

Rotation sets of toral homeomorphisms

Pointwise: difficult to work with. Invariant measures: too weak.

Misiurewicz-Ziemian rotation set

$$\rho(\widetilde{f}) = \Big\{ \lim_{k\to\infty} (\widetilde{f}^{n_k}(z_k) - z_k)/n_k : z_k \in \mathbb{R}^2,\ n_k \to \infty \Big\}$$
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Proposition

$\rho_{\mathrm{erg}}(\widetilde{f}) \subset \rho_p(\widetilde{f}) \subset \rho(\widetilde{f}) = \rho_m(\widetilde{f})$ (dimension 2!)

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Proof: $\rho_m(\widetilde{f}) = \mathrm{Conv}(\rho_{\mathrm{erg}}(\widetilde{f}))$ and $\rho(\widetilde{f})$ is convex $\implies \rho(\widetilde{f}) \supset \rho_m(\widetilde{f})$.

Convexity of the Misiurewicz-Ziemian rotation set

$$Q = (0,1)^2 \text{ unit square}, \quad \rho(\widetilde{f}) = \bigcap_{n\geqslant 0} \overline{\bigcup_{m\geqslant n} \frac{1}{m}\widetilde{\Delta}^m(Q)}$$

Follows from:

$\widetilde{f}^m(Q)$ is $\sqrt{2}$-**quasi-convex**: $\mathrm{Conv}(\widetilde{f}^m(Q)) \subset B_{\sqrt{2}}(\widetilde{f}^m(Q))$
($\Longrightarrow$ $\frac{1}{m}\widetilde{\Delta}^m_{\widetilde{f}}(Q) = \{(\widetilde{f}^m(z) - z)/m : z \in Q\}$ is $2\sqrt{2}/m$-quasi-convex.)

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Lemma (Quasi-convexity)

If $W \subset \mathbb{R}^2$ top. disk and $W \cap (W + v) = \emptyset$ for all $v \in \mathbb{Z}^2_*$, then W is $\sqrt{2}$-quasi-convex, i.e. $\mathrm{Conv}(W) \subset B_{\sqrt{2}}(W)$.

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Key lemma (Douady): If γ is a simple arc in $\mathbb{R}^2$ joining x to y and disjoint from the line segment L_{xy} and $v \in \mathbb{R}^2$ is such that $x + v \in D =$ disk bounded by $\gamma \cup L_{xy}$, then $\gamma \cap (\gamma + v) \neq \varnothing$. **(Blackboard.)**

Examples

Rigid rotations

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Rotation interval

Let $\phi(x) = |\sin(2\pi x)|$.

- $\widetilde{f}_1(x,y) = (x, y + \phi(x)) \quad \Longrightarrow \quad \rho(\widetilde{f}_1) = \{0\} \times [0,1].$
- $\widetilde{f}_2(x,y) = (x + \phi(y), y) \quad \Longrightarrow \quad \rho(\widetilde{f}_2) = [0,1] \times \{0\}.$

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Rotation set with interior

$$\widetilde{f} = \widetilde{f}_2 \circ \widetilde{f}_1 \quad \Longrightarrow \rho(\widetilde{f}) = [0,1] \times [0,1].$$

Rotation sets with nonempty interior: periodic points

A rational point $v/q \in \mathbb{Q}^2$ is **realized by a periodic point** if there is $z \in \mathbb{R}^2$ such that $\widetilde{f}^q(z) = z + v$.

Theorems (Franks '89)

- Every rational **extremal** point of $\rho(\widetilde{f})$ is realized by a periodic point;
- Every rational point in the **interior** of $\rho(\widetilde{f})$ is also realized.

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Note: Non-extremal boundary points may not be realized.

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Theorem (Misiurewicz, Ziemian '89)

Every element of $\mathrm{int}(\rho(\tilde{f}))$ is realized by an ergodic measure (in fact by a compact invariant set).

In particular, $\mathrm{int}(\rho(\tilde{f})) \subset \rho_p(\tilde{f})$.

Rotation sets with nonempty interior: entropy

Theorem (Llibre, Mackay '91)

If $\rho(\widetilde{f})$ has nonempty interior, then f has positive topological entropy.

Main tool: Thurston classification, Handel's global shadowing.

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Theorem (Nielsen-Thurston)

Let $g\colon S \to S$ be a homeomorphism of a compact surface. Then g is **isotopic** to a homeomorphism Φ of one of these types:

- Periodic: $\Phi^k = \mathrm{Id}$ for some k;
- Pseudo-Anosov:
  - There exist transverse measured foliations $\mathcal{F}^s$, $\mathcal{F}^u$ with (finitely many) singularities and $\lambda > 1$ such that $\Phi(\mathcal{F}^s) = \lambda^{-1}\mathcal{F}^s$, $\Phi(\mathcal{F}^u) = \lambda\mathcal{F}^u$.
- Reducible:
  - There is an invariant finite union of non-periphereal ess. simple loops.

Rotation sets with nonempty interior: entropy

Pseudo-Anosov maps

- They are transitive, have dense periodic points.
- Markov partitions, positive entropy.
- Stability properties (Handel '85): g isotopic to Φ (pA-map) $\Longrightarrow$
  - $\exists Y \subset\subset S$ and $h\colon Y \to S$ continuous surjection, $hf|_Y = \Phi h$.
  - $h_{top}(\Phi) \leqslant h_{top}(f)$.
  - Periodic points of Φ "persist" by isotopy.

Similar results for surfaces with finitely many punctures (marked points).

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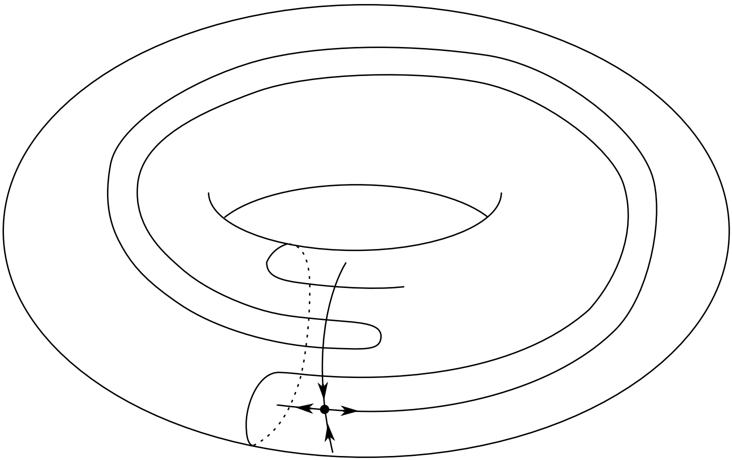
Proof of $\rho(\widetilde{f})$ with interior $\implies$ entropy

- Enough to prove it for f^n. We may assume $\rho(\widetilde{f})$ contains a large ball.
- fixed points x_1, x_2, x_3 with non-collinear rotation vectors $v_1, v_2, v_3 \in \mathbb{Z}^2$.
- f is isotopic to a (relative) pseudo-Anosov map rel $\{x_1, x_2, x_3\}$.

Example



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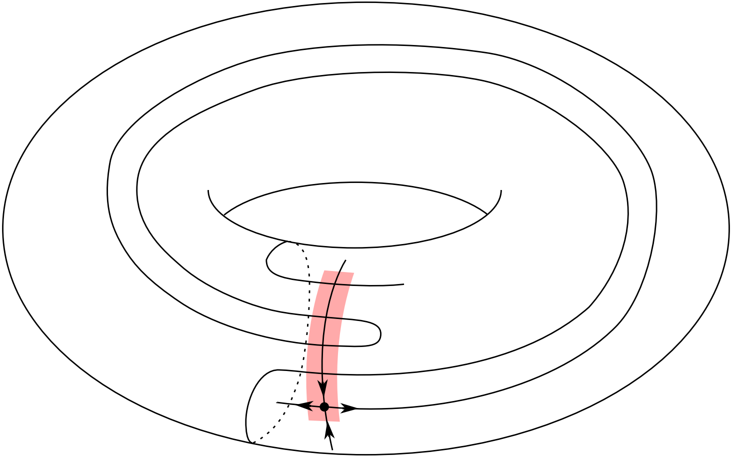

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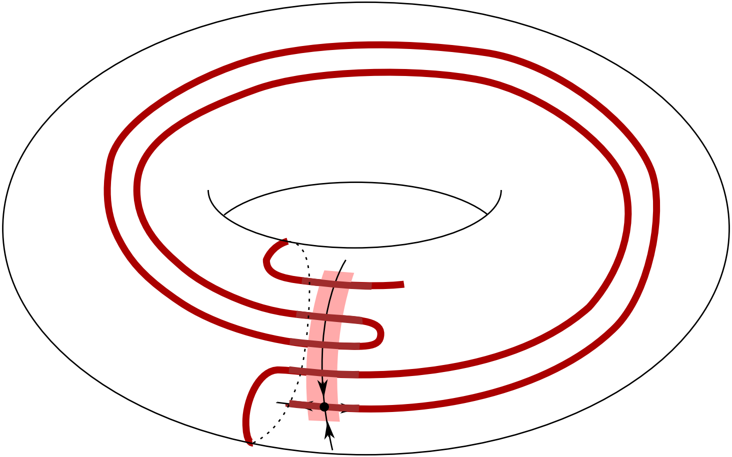

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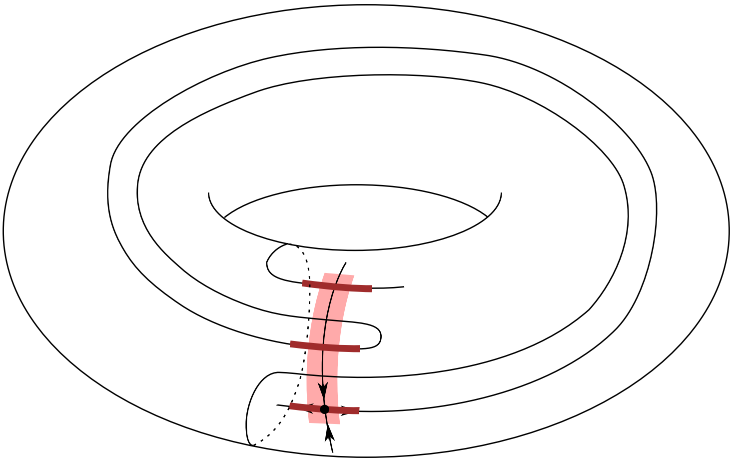

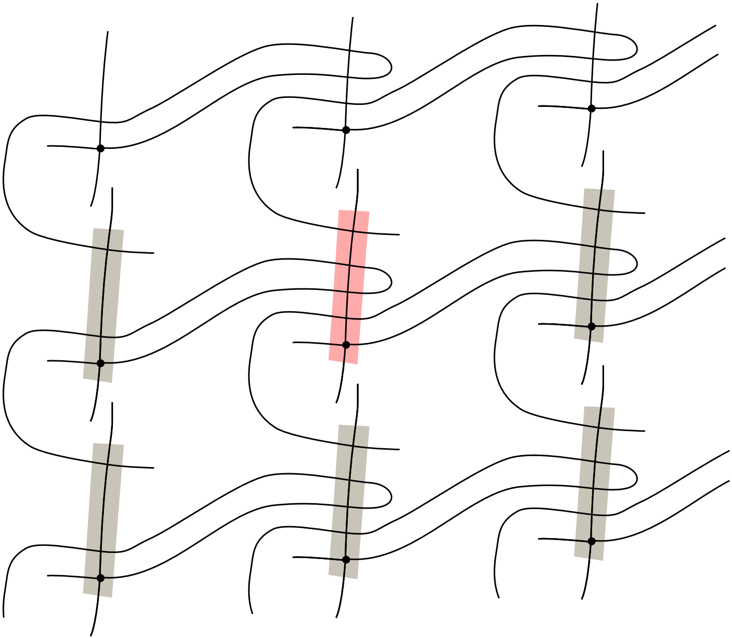

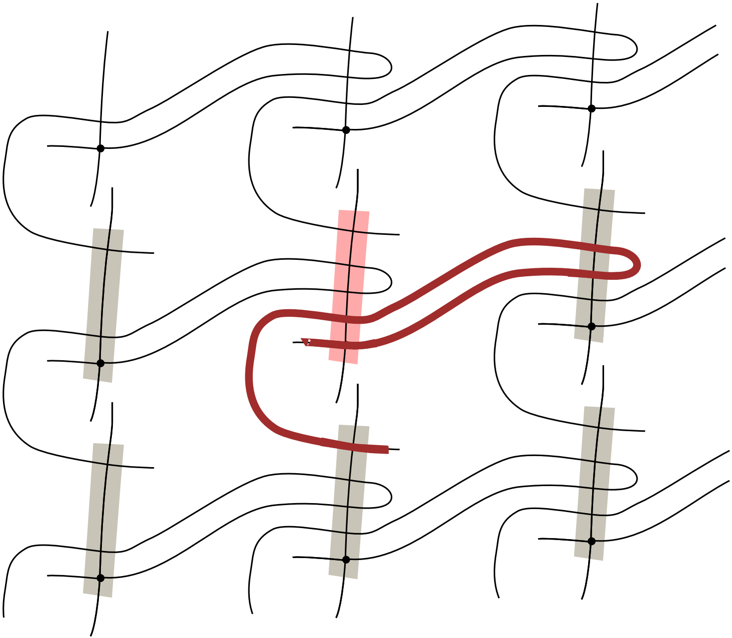

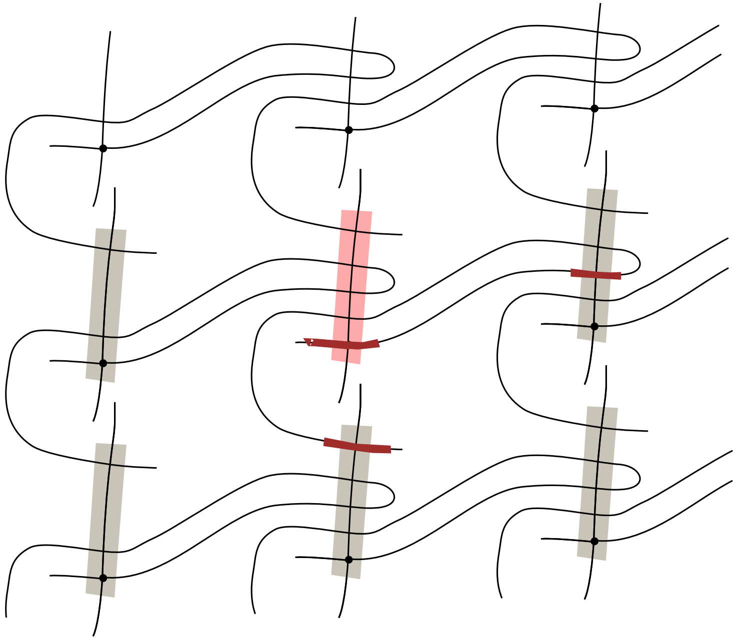

Rotation sets with nonempty interior

Theorem (Addas-Zanata 2013, 2015)

If f is $C^{1+\alpha}$ and $(0,0) \in \operatorname{int} \rho(\widetilde{f})$, then there exists a hyperbolic periodic point p for $\widetilde{f}$ such that for all $v \in \mathbb{Z}^2$, the stable manifold of p has a topologically transverse intersection with the unstable manifold of $p + v$ (and vice-versa).

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Theorem (Guelman, K., Tal, 2013)

If f is transitive and $(0,0) \in \operatorname{int} \rho(\widetilde{f})$, then $\widetilde{f}$ is transitive.

Rotation sets with nonempty interior

Theorem (Addas-Zanata 2013, 2015)

If f is $C^{1+\alpha}$ and $(0,0) \in \operatorname{int} \rho(\widetilde{f})$, then there exists a hyperbolic periodic point p for $\widetilde{f}$ such that for all $v \in \mathbb{Z}^2$, the stable manifold of p has a topologically transverse intersection with the unstable manifold of $p + v$ (and vice-versa).

Theorem (Guelman, K., Tal, 2013)

If f is transitive and $(0,0) \in \operatorname{int} \rho(\widetilde{f})$, then $\widetilde{f}$ is transitive.

Theorem (de Carvalho, K., Tal)

If f is $C^{1+\alpha}$ and the rotation set has nonempty interior, then f is monotonely semiconjugate to a “model map” which is: transitive, with dense periodic points, continuum-wise expansive , and more.

Shapes of rotation sets with empty interior

Question

Which compact convex sets are realizable as rotation sets?

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Which compact convex sets are realizable as rotation sets?

With empty interior (intervals)

- Single points (rotations);
- intervals with rational slope containing rational points;
- intervals with irrational slope and one rational endpoint (Katok);

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Can an interval of rational slope without periodic points be realized?

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Some advances [K., Passeggi, Sambarino 2016], [Kocsard 2016].

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- Convex polygons with rational vertices (Kwapisz '92)
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The set of extremal points is totally disconnected.

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The set of extremal points is totally disconnected.

The only compact convex sets known to be non-realizable are those whose boundary contain an interval of irrational slope with a rational non-extremal point (Le Calvez, Tal 2016)

Stability of the rotation set

Continuity

$\widetilde{f} \mapsto \rho(\widetilde{f})$ is not continuous, but it is upper-semicontinuous.

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The rotation set is stable if it does not change under small perturbations of the dynamics.

- Addas-Zanata 2004: C^0-stable $\implies$ rational extremal points;
- Guihéneuf 2016: Also C^1.
- Passeggi 2014: C^0-generically, stable + polygonal.
- Guihéneuf, K. 2016: Same thing area-preserving + estimates.

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We do not know any result about C^r-stability, or how to control the result even for C^0 perturbations. Related: C^r-enerically $\exists$ periodic point?

Mean rotation vectors in the area-preserving case

Let $\mu =$ Lebesgue measure on $\mathbb{T}^2$.

Lemma

If f, g are area-preserving, then $\rho(\widetilde{f}\widetilde{g}, \mu) = \rho(\widetilde{f}, \mu) + \rho(\widetilde{g}, \mu)$

Note: provides a group homomorphism $\mathrm{Diff}^r_\mu(\mathbb{T}^2) \to \mathbb{T}^2$.

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Theorem (Conley-Zehnder 83, Franks 88, Le Calvez 98)

If $\rho(\widetilde{f}, \mu) = (0, 0)$ then $\widetilde{f}$ has a fixed point.

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Theorem

For area-preserving diffeomorphisms, C^r-generically the rotation set has nonempty interior (any r).

Proof: Use perturbations of the form $R_v \circ f$ with v small.

Rotational deviations: the case with nonempty interior

No "sublinear" behavior:

Theorem

If $\rho(\widetilde{f})$ has nonempty interior, then there exists $M > 0$ such that

$$\forall n \in \mathbb{Z}, \quad \{\widetilde{f}^n(z) - z : z \in [0,1]^2\} = \Delta^n(\mathbb{T}^2) \subset B_M(n\rho(\widetilde{f})).$$

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- Dávalos 2013: rational polygons (BLC foliations, "forcing")
- Addas-Zanata 2015: $C^{1+\alpha}$ (Pesin theory, homoclinic intersections)
- Tal-Le Calvez 2016: general (BLC foliations, forcing theory)

Rotational deviations: the case with empty interior

Assume $\rho(\widetilde{f})$ has empty interior. If $\rho(\widetilde{f}) = \{v\}$ we call f a **pseudo-rotation**.

- $v \in \mathbb{R}^2 \backslash \mathbb{Q}^2$ (irrational pseudo-rotation).
  - Dynamics is aperiodc.
  - May be topologically weak-mixing, or even mixing (Kochergin)
  - May have unbounded rotational deviations (Kocsard, K.)
  - May have positive entropy (but not if f is smooth) (Rees, Katok)
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  - All of this may happen for area-preserving maps, minimal.
- $v = (p_1/q, p_2/q) \in \mathbb{Q}^2$ (rational pseudo-rotation)
  - Must have periodic points;
  - Interesting case: $v = (0,0)$ (take $\widetilde{g} = \widetilde{f}^q - (p_1, p_2)$).
  - If $\rho(\widetilde{f}) = (0,0)$ we say f is **irrotational**.
  - May have unbounded rotational deviations.
  - Katok's example.
  - Interesting case: f area-preserving.

Irrotational area-preserving homeomorphisms

Theorem (Lifted Poincaré recurrence) [K., Tal 2015]

If f is area-preserving and irrotational, then a.e. $z \in \mathbb{R}^2$ is $\widetilde{f}$-recurrent.

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Theorem (Le Calvez, Tal 2016)

If f is area-preserving and irrotational, then either $\mathrm{Fix}(f)$ is essential or the displacement is uniformly bounded: $\sup_{z\in\mathbb{T}^2, n\in\mathbb{Z}} \|\Delta^n(z)\| < \infty$.

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Question: Does the lifted Poincaré recurrence hold for irrotational area-preserving homeomorphisms of arbitrary surfaces?

Bounded deviations when $\rho(\widetilde{f})$ is an interval

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Yes if $\rho(\widetilde{f})$ is an interval with rational slope intersecting $\mathbb{Q}^2$. (Model: Vertical interval through the origin.)

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Theorem (Kocsard, 2016)

Yes in general, if f is minimal.

General case?