Rotation theory

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Dynamics of homeomorphisms of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$

Poincaré

For $f \in \mathrm{Homeo}_+(\mathbb{T})$, quantify the "average rotation"?

Rotation number

- Dynamical invariant:

$$\rho(f) \in \mathbb{R}/\mathbb{Z}, \quad \rho(hfh^{-1}) = \rho(f)\ \forall h \in \mathrm{Homeo}_+(\mathbb{T})$$

- Contains a lot of dynamical information:
  - $\rho(f) = p/q \in \mathbb{Q}/\mathbb{Z} \implies$ periodic orbit (of period q), and limit sets of all orbits are periodic orbits of the same type.
  - $\rho(f) \notin \mathbb{Q}/\mathbb{Z} \implies$ unique minimal set, unique ergodicity, monotone semiconjugation to irrational rotation, and if f is C^2 then f then actual conjugation (Denjoy)
- $f \mapsto \rho(f)$ is continuous.

Lifts

It is not obvious how to define $\rho(f)$ using f directly. We use lifts.

- $\pi : \mathbb{R} \to \mathbb{T}$ quotient projection (universal cover)
- π is a local homeomorphism, $\pi(x + k) = \pi(x)$ for all $k \in \mathbb{Z}$
- Given $f \in \mathrm{Homeo}_+(\mathbb{T})$, there is a lift $\tilde{f} : \mathbb{R} \circlearrowleft$ such that $\pi \tilde{f} = f \pi$:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\tilde{f}} & \mathbb{R} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbb{T} & \xrightarrow{f} & \mathbb{T} \end{array}$$

- $\tilde{f}(x + k) = \tilde{f}(x) + k$ for all $k \in \mathbb{Z}$
- Lifts are not unique, but they differ by a constant integer.

Rotation number

$$\rho(\tilde{f}, z) := \lim_{n\to+\infty} \frac{\tilde{f}^n(z) - z}{n}$$

- $\rho(\tilde{f}, z) = \rho(\tilde{f}, z + k)$ for all $k \in \mathbb{Z}$;
- For $x \in \mathbb{T}$, define $\rho(\tilde{f}, x) = \rho(\tilde{f}, z)$ where $z \in \pi^{-1}(x)$;
- $\rho(\tilde{f}^n, z) = n \cdot \rho(\tilde{f}, z)$;
- $\rho(\tilde{f} + k, z) = \rho(\tilde{f}, z) + k$;
- If $\tilde{f}_1, \tilde{f}_2$ are two lifts of f, then $\rho(\tilde{f}_1, z) - \rho(\tilde{f}_2, z) \in \mathbb{Z}$.

Theorem

The number $\rho(\tilde{f}, z)$ exists and is independent of z.

Rotation number of f

- $\rho(\tilde{f})$ is defined as $\rho(\tilde{f}, z)$ for any z;
- $\rho(f) = \pi(\rho(\tilde{f})) \in \mathbb{T}^1$ depends only of f (and not of the lift).

Rotation number

Displacement function:

- $\tilde{\Delta} : \mathbb{R} \to \mathbb{R}$: $\quad \tilde{\Delta}(\tilde{x}) = \tilde{f}(\tilde{x}) - \tilde{x}$ ($\mathbb{Z}$-periodic);
- $\Delta : \mathbb{T} \to \mathbb{R}$: $\quad \Delta(x) = \tilde{\Delta}(\tilde{x})$ for $\tilde{x} \in \pi^{-1}(x)$
- $\Delta^n(x) = \Delta_{\tilde{f}^n}(x) = \tilde{f}^n(\tilde{x}) - \tilde{x} = \sum_{k=0}^{n-1} \Delta(f^k(x))$ (Birkhoff sum);

Existence of $\rho(f, z)$ for some z

- The rotation number is a limit of Birkhoff averages: if $\pi(\tilde{x}) = x$,

$$\frac{\tilde{f}^n(\tilde{x}) - \tilde{x}}{n} = \frac{1}{n}\Delta^n(x) = \frac{1}{n}\sum_{k=0}^{n-1} \Delta(f^k(x)).$$

- Krylov-Bogoliubov: there exist f-invariant probabilities;
- Ergodic Theorem: the limit exists μ-a.e. (μ an f-invariant probab.);
- Moreover $\rho(\tilde{f}, x) = \int_{\mathbb{T}} \Delta \, d\mu$, μ-a.e. x;

Rotation number

Bounded Oscillation Property

$$|\Delta^n(x) - \Delta^n(y)| \leqslant M \quad \forall x,\, y,\, n$$

Since

$$\rho(\tilde{f}, x) = \lim_{n\to\infty} \frac{1}{n}\Delta^n(x)$$

exists for some x, BOP $\implies$ $\rho(\tilde{f}, x)$ exists for all x (and same value)

Bounded Rotational Deviations Property

$$|\Delta^n(x) - n\rho| \leqslant M \quad \forall x,\, n$$

In the large scale, the dynamics is close to the rigid rotation by $\rho(f)$.

Realization by periodic points

$$\rho(\tilde{f}) = p/q \iff \exists z \in \mathbb{R} \text{ s. t. } \tilde{f}^q(z) = z + p$$

Rotation number: Alternative approach

- Let (f_t) be an isotopy from $f_0 = \mathrm{Id}$ to $f_1 = f$ (model case: a flow);
- For $x \in \mathbb{T}$, the path $\gamma_x(t) = f_t(x)$ goes from x to $f(x)$;
- The path $\gamma_x^n = \gamma_x * \gamma_{f(x)} * \cdots * \gamma_{f^{n-1}(x)}$ goes from x to $f^n(x)$;
- Measure the average rotation of this path, and take the limit:

For each n take a lift $\tilde{\gamma}_x^n$ to $\mathbb{R}$ of the path γ_x^n, and

$$r(x) = \lim_{n\to\infty} \frac{\tilde{\gamma}_x^n(n) - \tilde{\gamma}_x(0)}{n}$$

This is independent of the lifts used, but depends on the chosen isotopy.

- If we change the isotopy, the number $r(x)$ changes by an integer.

In fact, (f_t) lifts to an isotopy $(\tilde{f}_t)$ starting from $\tilde{f}_0 = \mathrm{Id}_{\mathbb{R}}$, and the map $\tilde{f} := \tilde{f}_1$ is a lift of f such that $\rho(\tilde{f}, x) = r(x)$.

More general setting?

- Endomorphisms of the circle of degree 1;
- Homeomorphisms of the annulus;
- Flows on the torus;
- Homeomorphisms of the torus;
- Homeomorphisms of some topologically complicated compact connected sets;
- Homeomorphisms of arbitrary orientable surfaces;
- Heomomorphisms of general spaces which admit certain type of coverings.

Degree 1 endomorphisms of the circle

$f : \mathbb{T} \to \mathbb{T}$ continuous (maybe not invertible);

- $\tilde{f} : \mathbb{R} \to \mathbb{R}$ a lift $\implies \tilde{f}(x+1) = \tilde{f}(x) + m$ for some $m \in \mathbb{Z}$;
- $m = \deg(f)$, does not depend on the lift;
- $\deg(f) > 1 \implies$ rich dynamics (many periodic points, etc);

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- $\deg(f) = 1 \iff \tilde{\Delta} = \tilde{f} - \mathrm{Id}_{\mathbb{R}}$ is $\mathbb{Z}$-periodic;

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- $\deg(f) > 1 \implies$ rich dynamics (many periodic points, etc);
- $\deg(f) = 1 \iff \tilde{\Delta} = \tilde{f} - \mathrm{Id}_{\mathbb{R}}$ is $\mathbb{Z}$-periodic;
- Induces a displacement function $\Delta : \mathbb{T} \to \mathbb{R}$;
- The definition of $\rho(\tilde{f}, x)$ used before works the same way:

$$\rho(\tilde{f}, x) = \lim_{n\to\infty} \frac{1}{n}\Delta^n(x) = \lim_{n\to\infty} \frac{1}{n}\Delta(f^n(x))$$

- May not exist, or may depend on x.
- But exists for some x (as before, Birkhoff's theorem).

Degree 1 endomorphisms of the circle

$f : \mathbb{T} \to \mathbb{T}$ degree 1 map, $\tilde{f} : \mathbb{R} \to \mathbb{R}$ a lift.

Rotation set

$$\rho(\tilde{f}) = \{\lim_{n\to\infty} \rho(\tilde{f}, x) : x \in \mathbb{T},\ \text{limit exists}\} \subset \mathbb{R}$$

Rotation interval

$$\overline{\rho}(\tilde{f}) = [\rho_-(\tilde{f}), \rho_+(\tilde{f})] = [\inf \rho(\tilde{f}), \sup \rho(\tilde{f})]$$

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Invariant measures

- $\mathcal{M}(f) \subset \mathcal{M}(\mathbb{T})$ space of f-invariant probabilities;
- Mean rotation number of $\mu \in \mathcal{M}(f)$:
$$\rho(\tilde{f}, \mu) = \int_{\mathbb{T}} \Delta \, d\mu$$
- Ergodic theorem: $\rho(\tilde{f}, x)$ exists μ-a.e. and $\rho(\tilde{f}, \mu) = \int_{\mathbb{T}} \rho(\tilde{f}, x) \, d\mu(x)$;
- If μ is ergodic, $\rho(\tilde{f}, x) = \rho(\tilde{f}, \mu)$ for μ-a.e. x.

Recall: $\mathcal{M}(\tilde{f})$ is convex, ergodic measures = extremal points of $\mathcal{M}(\tilde{f})$.

Degree 1 endomorphisms of the circle

Theorem

Extremal rotation numbers $\rho_\pm$ *are mean rotation numbers of ergodic measures. In particular*

$$\rho_\pm(\tilde{f}) \in \rho(\tilde{f})$$

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Theorem (Ito '81)

The rotation set coincides with the rotation interval: $\overline{\rho}(\tilde{f}) = \rho(\tilde{f})$.

- Rational elements of $\overline{\rho}(\tilde{f})$ are realized by periodic points: if $p/q \in \overline{\rho}(\tilde{f})$ then there exists $z \in \mathbb{R}$ such that $\tilde{f}^q(z) = z + p$ (Exercise).
- The rotation set varies continuously.
- $\rho(\tilde{f})$ nonsingular interval $\implies$ topological entropy.

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Example: if $\tilde{f}(x) = x + \sin(2\pi x)$, then $\rho(\tilde{f}) = [-1, 1]$.

Homeomorphisms of the annulus

$$\mathbb{A} = \mathbb{T}^1 \times [0,1], \quad \pi : \tilde{\mathbb{A}} := \mathbb{R} \times [0,1] \to \mathbb{A}$$

$f : \mathbb{A} \to \mathbb{A}$ homeomorphism homotopic to Id, $\tilde{f} : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ lift.

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- $\tilde{f}$ commutes with $T\colon (x, y) \mapsto (x + 1, y)$;
- Horizontal displacement: $\tilde{\Delta}\colon \tilde{\mathbb{A}} \to \tilde{\mathbb{R}}$, $\tilde{\Delta}(\tilde{z}) = \mathrm{pr}_1(\tilde{f}(\tilde{z}) - \tilde{z})$
- $\tilde{\Delta}$ is T-periodic and induces $\Delta\colon \mathbb{A} \to \mathbb{R}$;

Rotation number of $z \in \mathbb{A}$

$\rho(\tilde{f}, z)$ is the limit as $n \to \infty$ (if $\exists$) of

$$\frac{1}{n}\Delta^n(z) = \frac{1}{n}\sum_{k=0}^{n-1} \Delta(f^n(z)) = \mathrm{pr}_1\left(\frac{\tilde{f}^n(\tilde{z}) - \tilde{z}}{n}\right)$$

Homeomorphisms of the annulus

- Rotation set (pointwise): $\rho(\tilde{f}) = \{\rho(\tilde{f}, z) : z \in \mathbb{A}, \text{ exists } \}$.
- Rotation interval: $\overline{\rho} = [\rho_-(\tilde{f}), \rho_+(\tilde{f})]$.
- Mean rotation number: $\rho(\tilde{f}, \mu) = \int_{\mathbb{A}} \Delta d\mu$ for $\mu \in \mathcal{M}(f)$.

As before:

- $\rho(\tilde{f}, z)$ exists μ a.e. and $\rho(\tilde{f}, \mu) = \int_{\mathbb{A}} \rho(\tilde{f}, z) d\mu(z)$
- μ ergodic $\implies \rho(\tilde{f}, z) = \rho(\tilde{f}, \mu)$ for μ-a.e.
- $\rho_\pm$ correspond to ergodic measures.

New situation

(1) $\rho(\tilde{f})$ may fail to be an interval ($\rho(\tilde{f}) \subsetneq \overline{\rho}(\tilde{f})$);

(2) Rational elements of $\overline{\rho}(\tilde{f})$ may fail to be realized by periodic points;

(3) No continuous dependence (but upper-semicontinuous OK).

(4) $\rho(\tilde{f}, z)$ may fail to exist for some $z's$.

Examples.

Homeomorphisms of the annulus

$f : \mathbb{A} \to \mathbb{A}$ isotopic to the identity ($f \in \mathrm{Homeo}_0(\mathbb{A})$)

Theorem (Poincaré-Birkhoff)

If f is area-preserving and has the boundary twist condition then f has a fixed point (in fact, at least two)

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If f is area-preserving and $0 \in \overline{\rho}(\tilde{f})$ for some lift then $\tilde{f}$ has a fixed point.

Corollary (Realization of rational points)

If f is area-preserving, every rational $p/q \in \rho(\tilde{f})$ is realized by a periodic point, i.e. there is $\tilde{z} \in \tilde{\mathbb{A}}$ such that $\tilde{f}^q(\tilde{z}) = \tilde{z} + (p, 0)$.

Birkhoff/Kerekjarto: area-preserving $\leftrightarrow$ "curve intersection property"

Homeomorphisms of the annulus

Theorem (Handel)

The (pointwise) rotation set is closed. Moreover, for every $\alpha \in \rho(\tilde{f})$

(a) *α is realized by an ergodic measure;*

(b) *if α is rational, it is realized by a periodic orbit;*

(c) *if α is not in an exceptional discrete set Q, it is realized by a compact invariant set.*

Part (c) means that there is a compact invariant set $K_\alpha \subset \mathbb{A}$ such that $\rho(\tilde{f}, x) = \alpha$ for all $x \in K_\alpha$.

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Question

Is the exceptional set empty?

The proof is convoluted and relies on Nielsen-Thurston classification and global shadowing properties of pseudo-Anosov maps.

Area-preserving homeomorphisms of the annulus

With the area-preserving (or nonwandering) hypothesis, one gets the same results that hold for degree 1 endomorphisms of the circle:

Corollary (Franks + Handel)

If f is area-preserving then the pointwise rotation set is a closed interval i.e. $\rho(\tilde{f}) = \overline{\rho}(\tilde{f})$. Moreover, every rational element is realized by a periodic orbit and every element is realized by an ergodic measure.

Theorem (Le Calvez)

If f is area-preserving and C^1, then the exceptional set is empty, except perhaps for the endpoints of $\rho(\tilde{f})$.

A basic result from Brouwer theory

Fixed point index of a loop $\gamma \subset \mathbb{R}^2$

If $f \in \mathrm{Homeo}_+(\mathbb{R}^2)$ is such that $\gamma \cap \mathrm{Fix}(f) = \varnothing$, the (Lefschetz) *fixed point index* of C is the number $I(f, \gamma) \in \mathbb{Z}$ defined as the degree of the map $\theta \colon \mathbb{S}^1 \to \mathbb{S}^1$ defined by

$$\theta \mapsto \frac{f(\gamma(\theta)) - \gamma(\theta)}{\|f(\gamma(\theta)) - \gamma(\theta)\|}.$$

where $\gamma \colon \mathbb{S}^1 \to \mathbb{R}^2$ is a parametrization of γ.

- The index is invariant by homotopy of γ in $\mathbb{R}^2 \backslash \mathrm{Fix}(f)$.
- If $I(f, \gamma) \neq 0$ then f has a fixed point in the disk bounded by γ.

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Lemma (Brouwer)

If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is an orientation-preserving homeomorphism which has a non-fixed periodic point, then there exists simple loop of index 1.

Thus, $\mathrm{Per}(f) \neq \emptyset \implies \mathrm{Fix}(f) \neq \emptyset$.

Free Disks Lemma

A set K is *free* (for f) if $f(K) \cap K = \varnothing$

Definition

A *free disk chain* is a sequence $D_0, \ldots, D_n$ of open topological disks s.t.

- The disks are pairwise disjoint and free for f;
- For each i there is $k_i > 0$ such that $f^{k_i}(D_i) \cap D_{i+1} \neq \varnothing$.

If $D_n = D_0$ we say (D_i) is a *periodic free disk chain*.

Free Disks Lemma

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Free Disks Lemma / Franks' Lemma

If $f \in \mathrm{Homeo}_+(\mathbb{R}^2)$ has a periodic free disk chain, then f has an index 1 simple loop (hence a fixed point).

Follows from Brouwer's lemma using the following fact:

If (f_t) is an isotopy s.t. $\forall t$, $\mathrm{Fix}(f_t) \cap \gamma = \emptyset$, then $I(f_0, \gamma) = I(f_1, \gamma)$.

Brouwer homeomorphisms

Corollary

If $f \in \mathrm{Homeo}_+(\mathbb{R}^2)$ has a recurrent point, or even a nonwandering point, then f has a fixed point.

A Brouwer homeomorphism is an $f \in \mathrm{Homeo}_+(\mathbb{R}^2)$ without fixed points. Brouwer homeomorphisms have no wandering points.

Lemma

If K is a compact connected set and f is a Brouwer homeomorphism such that $f(K) \cap K = \emptyset$, then $f^n(K) \cap K = \emptyset$ for all $n \neq 0$.

Otherwise, one may produce a periodic free disk chain.

Remark

Brouwer's Plane Translation Theorem says that moreover every point belongs to a "Brouwer line" (more on that later).

Realizing periodic points

Theorem (Franks)

If $\tilde{f}$ is a lift of an area-preserving $f \in \mathrm{Homeo}_0(\mathbb{A})$, every rational $p/q \in \overline{\rho}(\tilde{f})$ is realized by a periodic point of f, i.e. there exists $z \in \tilde{\mathbb{A}}$ such that $\tilde{f}^q(z) = z + (p, 0)$.

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Reduction to $p/q = 0$

$$p/q \in \rho(\tilde{f}) \iff 0 \in \rho(\tilde{f}^q - (p, 0))$$

Thus it suffices to show that $0 \in \overline{\rho}(\tilde{f}) \implies \mathrm{Fix}(\tilde{f}) \neq \emptyset$.

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Idea: Assuming that $\mathrm{Fix}(\tilde{f}) = \emptyset$, show that $\tilde{f}$ has a periodic free disk chain (contradiction).

Realizing periodic points

Assume $0 \in \overline{\rho}(\tilde{f})$ and $\mathrm{Fix}(\tilde{f}) = \varnothing$. Find a periodic free disk chain.

Positively and negatively returning disks

A positively (negatively) returning disk $D \subset \tilde{\mathbb{A}}$ is a free disk such that $\tilde{f}^n(D) \cap (D + (k, 0)) \neq 0$ for some $k \geqslant 0$ ($k \leqslant 0$) and $n > 0$.

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- $\overline{\rho}(\tilde{f}) = [\rho_-, \rho_+]$ with $\rho_- \leqslant 0 \leqslant \rho_+$;
- There exist an ergodic $\mu_\pm$ such that $\rho(\tilde{f}, \mu_\pm) = \rho_\pm$;

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- $\overline{\rho}(\tilde{f}) = [\rho_-, \rho_+]$ with $\rho_- \leqslant 0 \leqslant \rho_+$;
- There exist an ergodic $\mu_\pm$ such that $\rho(\tilde{f}, \mu_\pm) = \rho_\pm$;
- Fix two small free disks $D^\pm$ such that $\mu_\pm(D^\pm) > 0$;

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Lemma

If $\tilde{f}$ has positively returning free disk D^+ and a negatively returning free disk D^-, then $\tilde{f}$ has a periodic free disk chain.

- $\overline{\rho}(\tilde{f}) = [\rho_-, \rho_+]$ with $\rho_- \leqslant 0 \leqslant \rho_+$;
- There exist an ergodic $\mu_\pm$ such that $\rho(\tilde{f}, \mu_\pm) = \rho_\pm$;
- Fix two small free disks $D^\pm$ such that $\mu_\pm(D^\pm) > 0$;
- Show that D^+ is positively returning and D^- is negatively returning.
- (Use first return map + Kac's lemma).