Minimal Systems on Cantor Set

Maryam Hosseini

Thematic Program in Dynamical Systems

IPM, Tehran

February 2017

Outline

1. Introduction
2. Examples:
   - Odometers
   - Substitutions
   - Toeplitz
3. Kakutani-Rokhlin towers for minimal systems
   - Bratteli diagrams and Vershik Systems with Examples
   - Some Dynamical Properties of Vershik Homeomorphisms

Introduction

Minimal systems: Natural generalizations of periodic orbits and topological analogous of ergodic systems, defined by [G. D. Birkhof, 1912].

Extension to Cantor set:

Theorem. (P. Alexandroff, 1927)

Every compact metric space is a continuous image of the Cantor set.

Let (X, T) be a minimal system on a compact metric space.

$$\exists\ F : C \to X, \quad C \text{ is the Cantor set}$$

which is continuous and onto. Set

$$K := \{(x_n)_{n\in\mathbb{Z}};\ x_n \in C,\ F(x_{n+1}) = T(F(x_n))\}.$$

$$K = \{(x_n)_{n\in\mathbb{Z}};\ x_n \in C,\ F(x_{n+1}) = T(F(x_n))\} \subseteq C^{\mathbb{Z}}$$

and is σ-invariant. Let Z be a minimal subset of (K, σ). Then

$$\psi : (Z, \sigma) \to (X, T)$$

$$\psi((z_n)_{n\in\mathbb{Z}}) = z_0$$

makes the factoring.

Remark. Note that Z is a closed subset of the $C^{\mathbb{Z}}$ and so is a Cantor set.

1. Odometers (adding Machines)

Let $J = (j_1, j_2, \cdots)$ be a sequence of natural numbers and

$$X = \{(x_n)_{n\in\mathbb{N}_0} : \quad 0 \le x_i \le j_i - 1\}.$$

The *adding machine* is defined by the map $T : X \to X$ with

$$T(x_0, x_1, \cdots) = (x_0, x_1, \cdots) + (1, 0, 0, \cdots).$$

The addition is component-wise with carrying to the right. This system is minimal and *distal,* means that

$$\forall x, y \in X,\ \exists \delta > 0; \quad d(T^n x, T^n y) > \delta,\ \forall n \ge 0.$$

In fact, $\delta_{x,y} = d(x, y)$. In fact, it is *equicontinuous*, means that $\{T^n\}_n$ is an equicontinuous family.

Theorem. (See [P. kurka 2003])

Every minimal equicontinuous system on Cantor set is conjugate to an odometer.

proof.
It suffices to consider the equivalent metric

$$d(x,\, y) = \sup_n d(T^n x, T^n y).$$

Corollary.

The maximal equicontinuous factor of a minimal distal system on Cantor set is conjugate to an odometer.

Let $n_i := j_i j_{i-1} \cdots j_1$. It's pretty clear that $T^{n_i} \to \mathrm{id}$, or

$$\forall x \in X, \quad T^{n_i} x \to x.$$

This property is called *rigidity* along the sequence $\{n_i\}_i$.

Proposition. (E. Glasner, D. Maon, 1975)

Any (infinite) minimal rigid system on Cantor set is conjugate to an odometer.

Proof. Exercise (Hint: show that it is equicontinuous).

Odometers are also called rotations or Kronecker system on Cantor set as they are isometries.

Odometers from algebraic point of view

Let $(p_i)_{i\geq 1}$ be a sequence of natural numbers that

$$\forall i \geq 1, \quad p_i \geq 2, \quad p_i | p_{i+1}.$$

Consider the following inverse limit system:

$$(\mathbb{Z}_{p_1}, \imath_1) \xleftarrow{\phi_1} (\mathbb{Z}_{p_2}, \imath_2) \xleftarrow{\phi_2} \cdots \longleftarrow (Z, \imath)$$

where $\imath_i(z) = z + 1 \pmod{p_i}$ and

$$Z = \{(z_n)_{n\in\mathbb{N}}; \quad z_n \in \mathbb{Z}_{p_n}, \quad \phi_i(z_i) = z_i \pmod{p_{i-1}}\}$$

and

$$\imath(z_1, z_2, \cdots) = (z_1, z_2, z_3, \cdots) + (1, 1, 1, \cdots).$$

Exercise. Show that $(Z, \imath)$ is conjugate to the odometer based on the sequence $(p_i/p_{i-1})_i$.

2. Substitutions

Let A be a set of alphabets, like $A = \{1, 2, \ldots, k\}$ and A^+ be the set of words with letters in A.
A *substitution* on A is a map $\tau : A \to A^+$ that

$$\forall a \in A, \ |\tau^n(a)| \to \infty.$$

By *concatenation*, one can extend such a map to A^+:

$$\forall w = w_1 w_2 \ldots w_k \in A^+, \ \tau(w) = \tau(w_1)\tau(w_2)\ldots\tau(w_k).$$

So $\tau^n : A \to A^+$ is also a substitution,

$$\forall a \in A, \quad \tau^n(a) = \tau^{n-1}(\tau(a)) = \cdots = \overbrace{\tau(\tau(\cdots(\tau}^{n\ times}(a)\cdots).$$

A substitution is *primitive* if

$$\forall a, b \in A, \ \exists p > 0; \ a \text{ appears in } \tau^p(b).$$

Fixed points of a substitution: $\{x \in X_\tau : \ \tau(x) = x\}$.

Example i) Let $A = \{0,1\}$ and $\tau(0) = 001$, $\tau(1) = 01$. Then

$$0 \overset{\tau}{\longmapsto} 001 \overset{\tau}{\longmapsto} 00100101 \overset{\tau}{\longmapsto} 0010010100100101001 01 \overset{\tau}{\longmapsto} \cdots ;$$

$$1 \overset{\tau}{\longmapsto} 01 \overset{\tau}{\longmapsto} 00101 \overset{\tau}{\longmapsto} 0010010100101 \overset{\tau}{\longmapsto} \cdots .$$

Example ii)(Thue-Morse) Let $A = \{0,1\}$ and $\tau(0) = 01$, $\tau(1) = 10$. Then

$$0 \overset{\tau}{\longmapsto} 01 \overset{\tau}{\longmapsto} 0110 \overset{\tau}{\longmapsto} 01101001 \overset{\tau}{\longmapsto} \cdots ,$$

$$1 \overset{\tau}{\longmapsto} 10 \overset{\tau}{\longmapsto} 1001 \overset{\tau}{\longmapsto} 10010110 \overset{\tau}{\longmapsto} \cdots .$$

Example iii) Let $A = \{0,1,2\}$. Then

$$0 \longmapsto 01, \quad 1 \longmapsto 2, \quad 2 \longmapsto 012$$

Example iv) Let $A = \{0,1\}$. Then

$$0 \longmapsto 010, \quad 1 \longmapsto 111.$$

If there exists at least one letter $a \in A$ so that $\tau(a)$ begins with a, then we have at least one fixed point.

Definition.

$$\forall x \in A^{\mathbb{Z}}, \quad \mathcal{L}(x) = \{u \in A^{+};\ \exists p > 0, u \prec \tau^{p}(x)\}.$$

It is easy to see that for a primitive τ,

$$x, y \in A,\ \tau(x) = x,\ \tau(y) = y \quad \Rightarrow \quad \mathcal{L}(x) = \mathcal{L}(y).$$

Definition.

A primitive substitution is proper if it has a unique fixed point.

Remark.

If $\exists r, \ell \in A$ such that $\forall a \in A,\ \tau(a)$ starts with r and ends with ℓ and $r\ell$ is admissible then τ is proper.

Substitution dynamical systems

Definition.

Let X_τ be a subset of $A^{\mathbb{Z}}$ associated to the language of the fixed points of a primitive τ, i.e.

$$X_\tau = \{x \in A^{\mathbb{Z}} : \ \forall i < j, \quad x_i x_{i+1} \cdots x_j \in \mathcal{L}(a); \ a = \tau(a)\}.$$

X_τ together with the restriction of the shift map σ is called a Substitution dynamical system, (X_τ, σ).

In other words, a subshift (X, σ) with the alphabet A, is a substitution if

$$\exists \text{ a primitive } \tau : A \to A^+, \ w = \tau(w); \ X_\tau = \overline{\{\sigma^n(w)\}_n},$$

Proposition. (F. Durand, B. Host, C. Skau, 1999)

Every substitution dynamical system is conjugate to the closure orbit of the fixed point of a proper substitution.

Systems associated to sequences

Let $u = (u_n)_n$ be a sequence in a shift space and set

$$X = \overline{\{\sigma^n(u)\}_n}.$$

Proposition. (See [M. Queffelece '87])

(X, σ) is minimal iff u is uniformly recurrent.

Recall that *uniform recurrence* means that for any words w the set of gaps between any two consecutive occurrences of w is bounded.

Corollary.

Every substitution dynamical system, (X, σ) is minimal.

Let $u = (u_n)_n$ be a sequence in a shift space and $\ell_B(C)$ be the number of occurrence of B in C, where B and C are two admissible words.
We say that u has *uniform word frequencies* if

$$\forall\ B : \lim_{n\to\infty} \frac{\ell_B(u_k\ \ldots u_{k+n})}{n+1}$$

exists uniformly in k (independent from k).

Proposition. (See [M. Queffelec '87])

(X, σ) associated to the sequence u is uniquely ergodic iff u has uniform word frequencies.

Hint. Use point-wise ergodic theorem.

The invariant measure

Corollary.

Every substitution dynamical system, (X, σ) is uniquely ergodic.

In fact, for the substitution system (X_τ, σ) with alphabet A, for every $a \in A$ the map μ defined by

$$\mu := \lim_{j\to\infty} \frac{1}{|\tau^j(a)|} \sum_{n<|\tau^j(a)|} \delta_{T^n u}$$

is an invariant Borel measure for the system which is unique.

Linear complexity

Proposition. (See [M. Queffelec '87])

Every substitution dynamical system has zero entropy.

Proof. Consider the *incidence matrix of the substitution.* Using Perron-Frobenius Theorem, for the fixed point u, there exists $r > 0$ such that

$$p_u(n) \leq rn \ \Rightarrow \ \lim_{n\to\infty} \frac{1}{n} \log(p_u(n)) = 0.$$

Example i) Sturmian systems are substitutions or generated by finitely many substitutions. These are almost one to one extensions of irrational rotations on the unit circle with $p_u(n) = n + 1$.

(weakly) mixing substitution

Example ii) Chacon's minimal weakly mixing and non-mixing substitution system (X, σ), where X is the orbit closure of the first fixed point of the following substitution:

$$0 \longmapsto 0010, \quad 1 \longmapsto 1,$$

which is non-primitive. But there exists a primitive substitution with 3 symbols that makes a conjugate system. Example iii)

Dekking's and Kean's topologically mixing substitution system coming from:

$$0 \longmapsto 001, \quad 1 \longmapsto 11100.$$

Remark. (Dekking, Kean, 1978)

A substitution can never be strongly mixing with respect to its unique invariant measure.

3. Toeplitz sequence, See [P. Kurka 2003]

A point x in dynamical system (X, T) is *quasi-periodic* if

$$\forall U \text{ open set } ; \; x \in U, \; \exists p > 0; \; T^{np}(x) \in U, \; \forall n \geq 1.$$

Recall that in odometers all points are quasi-periodic.

Definition.

A point $x \in A^{\mathbb{N}}$ is Toeplitz if there exists an increasing sequence $(p_i)_{i \geq 0}, p_i \in \mathbb{N}$ such that

- $p_i | p_{i+1}$,
- *for every $n \geq 0$ there exists some i so that $n \in per_{p_i}(x)$, where*

$$per_{p_i}(x) = \{k \in \mathbb{N} : \; \forall n \; x_{k+pn} = x_k\}.$$

So any Toeplitz sequence is quasi-periodic (w.r.t. shift map).

The p-skeleton of x, $S_p(x)$, is defined by

$$S_p(x) = \begin{cases} x_i & \text{if } i \in per_p(x) \\ * & \text{if } i \notin per_p(x). \end{cases}$$

So to construct the toeplitz sequence we need the

$$(p_i)_{i\geq 0},\ r_i := \min\{k:\ k \in per_{p_i}(x)\}.$$

to find $S_{p_i}(x)$.

Example. Let $A = \{0,1\}$ and construct the toeplitz sequence with the periodic structure $(p_n)_n = (2^n)_{n\geq 1}$ and $r_2 = 0,\ \ r_4 = 1\ \ r_8 = 3,\ \ r_{16} = 7, \cdots$. Then

$$\begin{array}{lllllllllll} S_1(x) & = * & * & * & * & * & * & * & * & \cdots \\ S_2(x) & = 1 & * & 1 & * & 1 & * & 1 & * & \cdots \\ S_4(x) & = 1 & 0 & 1 & * & 1 & 0 & 1 & * & \cdots \\ S_8(x) & = 1 & 0 & 1 & 1 & 1 & 0 & 1 & * & \cdots \\ S_{16}(x) & = 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & \cdots \end{array}$$

Toeplitz dynamical systems, See [P. Kurka 2003]

Definition.

A subshift (X, σ) is Toeplitz system if $X = \overline{\{\sigma^n(x)\}}_{n\geq 0}$ where x is a Topelitz sequence.

Remark.

It is clear that a Toeplitz sequence is uniformly recurrent and so any Toeplitz system is minimal.

$$\text{regular Toeplitz} : \lim_{i\to\infty} \frac{|(S_{p_i}(x))_*|}{p_i} = 0.$$

$$\text{Toeplitz}\begin{cases} \text{regular} & \rightsquigarrow \quad \text{is uniquely ergodic} \\ \text{non-regular} & \rightsquigarrow \quad \text{is not necessarily uniquely ergodic.} \end{cases}$$

$$\text{Toeplitz}\begin{cases} \text{regular} & \rightsquigarrow \quad \text{has zero entropy} \\ \text{non-regular} & \rightsquigarrow \quad \text{entropy might be positive.} \end{cases}$$

Toeplitz and odometers

Proposition.

Any Toeplitz system is an almost one to one extension of an odometer.

Proof. Consider the periodic structure $\mathbf{p} = (p_i)_{i \geq 0}$ of the system and let

$$A_n^i := \overline{\{\sigma^{n+p_i m} x : \; m \in \mathbb{N}\}}, \; i > 0, \; 0 \leq n < p_i.$$

These are clopen subsets of X and

$$y \in A_n^i \iff S_{p_i}(y) = S_{p_i}(\sigma^n x).$$

Now define the map $\pi : X \to Z_{\mathbf{p}}$ by

$$(\pi(x))_i = n \;\text{ iff }\; x \in A_n^i.$$

It is not hard to see that π is continuous and $|\pi^{-1}(x)| = 1$ if x is Toeplitz. So π is almost one to one. □

Topological characterization

Definition. ((Jacob- Kean, 1969), (Eberlien1970), (Downarowisz-Lacorix 1998))

A dynamical system on a Cantor set is Toeplitz if it is

- *minimal;*
- *expansive;*
- *and almost one to one extension of an odometer.*

Note that the second condition can be replaced by *being subshift.*

Theorem. (P. Kurka 2003)

A Cantor system is conjugate to a subshift iff it is expansive.

Toeplitz and substitutions

A substitution with constant length and common prefix for all letters will make a Toeplitz sequence.

Example. Let $A = \{0, 1\}$ and define

$$\tau = \begin{cases} 0 \longmapsto 11 \longmapsto 1010 \longmapsto 10111011 \longmapsto \cdots \\ 1 \longmapsto 10 \longmapsto 1011 \longmapsto 10111010 \longmapsto \cdots . \end{cases}$$

The unique fixed point, x, of this substitution has 1 at all x_{2n}. Because $\tau(0)$ and $\tau(1)$ have common prefix 1. Similarly, satrting from x_1 and with period 4 there are 0's at all x_{4n+1} and so on. Therefore, x is a Toeplitz sequence.

A Tower for a Cantor minimal systems, [I. Putnam 1989]

Let (X, T) be a minimal Cantor system, $\mathcal{P}$ a finite (clopen) partition and Y be a non-empty clopen subset of X. Define $\lambda : Y \to \mathbb{Z}$ by

$$\lambda(y) := \inf\{n \geq 1 : \ T^n(y) \in Y\}, \quad y \in Y.$$

Suppose that

$$\lambda(Y) = \{J_1, J_2, \cdots, J_K\}.$$

For each $1 \leq k \leq K$, set $Y(k, j) := T^j(\lambda^{-1}(J_k))$. Then

- $\bigcup_{k=1}^{K} Y(k, 1) = T(Y)$;
- $T(Y(k, j)) = Y(k, j+1)$, for $1 \leq j \leq J_k$;
- $\bigcup_{k=1}^{K} Y(k, J_k) = Y$.

$\bigcup_{k,j} Y(k, j)$ is closed and T-invariant; so it covers X. Moreover, we can break the columns of $\mathcal{T}$ to have a refinement of $\mathcal{P}$. This is called a *Kakutani-Rokhlin* tower $\mathcal{T}$ for (X, T).

Nested Kakutani-Rokhlin towers.

Theorem.

For any Cantor minimal system (X, T) and $x_0 \in X$, there exists a nested sequence of Kakutani-Rokhlin towers $\{\mathcal{T}\}_{n\geq 0}$ whose intersection is $\{x_0\}$ and $\bigcup_{n\geq 0} \mathcal{T}_n$ generates the topology of X.

Proof. Let $\{\mathcal{P}_i\}_{i\geq 0}$, $\mathcal{P}_i \preceq \mathcal{P}_{i-1}$, be a sequence of finite clopen partitions of X whose union generates the topology on it. Choose a decreasing sequence of clopen subsets

$$Y_0 \supset Y_1 \supset Y_2 \supset \cdots$$

converging to $\{x_0\}$. By induction, there exists a sequence of towers

$$\mathcal{T}_n = \bigcup_{k=1}^{K} \bigcup_{j=1}^{J_k} (Y_n, j),\ n \in \mathbb{N}$$

such that $\mathcal{T}_n \prec \mathcal{P}_n$. $\square$

Example 1. Odometer

Consider $Z_{\mathbf{p}}$ with $\mathbf{p} = (2^n)_{n\geq 1}$ with alphabet $A = \{0, 1\}$. Let $x = (0,\ 1,\ x_2,\ \cdots)$ and $Y_1 = [01]$.Then $H_1 = \{4\}$ and

$$\mathcal{T}_1 := [01] \longmapsto [11] \longmapsto [00] \longmapsto [10].$$

Similarly, let $Y_2 = [01x_2] \subset Y_1$. Then $H_2 = 8$ and

$$\mathcal{T}_2 := [01x_2] \longmapsto \cdots \longmapsto [00(x_2+1)], \cdots \longmapsto [10x_2] \prec \mathcal{T}_1.$$

Therefore, at each step n the hight of the tower $\mathcal{T}_n$ is 2^n with the base $Y_n := [01x_2 \cdots x_{2^n-1}]$ which converge to x.

For general case, if the odometer is $Z_{\mathbf{p}}$ with $\mathbf{p} = (j_i)_{i\geq 1}$, for any arbitrary point x, there exists a sequence of towers with intersection equal to $\{x\}$ and at each step n the tower is a single column of height

$$H_n = j_n j_{n-1} \cdots j_1.$$

Example 2. primitive proper substitutions

Let $A = \{0, 1\}$ and $\tau(0) = 001$, $\tau(1) = 01$. Clearly

$$\mathcal{T}_0 = \{X\} = \{[0] \cup [1]\}.$$

So $\mathcal{T}_0$ has two columns each one with a single cell. Then

$$0 \xrightarrow{\tau} 001 \xrightarrow{\tau} 00100101 \xrightarrow{\tau} 0010010100100101001001 \xrightarrow{\tau} \cdots.$$

If a point x belongs to $[0]$ then two cases might be happened

- $x \in [00]$, then the first return time to $[0]$ for x is 3 because of 0010;
- or $x \in [01]$ which implies that the first return time to $[0]$ for x is 2 because of 010.

Consider $[0] = V_1 \cup V_2 \cup V_3$ and $[1] = W_1 \cup W_2$, we will have a tower $\mathcal{T}_1$ with two columns:

$$V_1 \longmapsto V_2 \longmapsto W_1,$$
$$V_3 \longmapsto W_2$$

that covers X.

To make a finer partition than $\mathcal{T}_1$, it is enough to consider two clopen sets:

$$U := V_1 \longmapsto V_2 \longmapsto W_1, \qquad Z := V_3 \longmapsto W_2$$

from $\mathcal{T}_1$.Since we had substitution map, again we have

$$U = U_1 \cup U_2 \cup U_3, \quad Z = Z_1 \cup Z_2.$$

And the movements between the cells are similarly repeated:

$$\begin{aligned} &U_1 \longmapsto U_2 \longmapsto Z_1, \\ &U_3 \longmapsto Z_2 \end{aligned}$$

which makes us a tower $\mathcal{T}_2$ with two columns that refines $\mathcal{T}_1$. An inductive argument will make the nested sequence of towers.

..., [F. Durand, B. Host, C. Skau '99]

In general, the Kakutani-Rokhlin towers for a substitution dynamical system (X_τ, σ), with alphabet A,

- have (at all the steps n), $|A|$ columns and for each $a \in A$ there exists a column of the height the height $|\tau(a)|$;
- *the order of the appearance of the columns of each tower $\mathcal{T}_{n-1}$ as the sub-columns of the next tower $\mathcal{T}_n$, is the same as $\mathcal{T}_0$'s appearing in $\mathcal{T}_1$.*

Note that at each step n the given finite clopen partition which is refined by $\mathcal{T}_n$ is the usual cylinder sets of the shift space with length 2^n.

Example 3. Toeplitz

Let (X, T) be a Toeplitz system which is the closure orbit of the Toeplitz sequence x with periodic structure $(p_i)_{i\geq 1}$. Recall that the clopen sets

$$A_n^i := \overline{\{\sigma^{n+p_i m}x :\ m \in \mathbb{N}\}}, \quad n < p_i, i > 0$$

have the following properties:

- $y \in A_n^i \iff S_{p_i}(y) = S_{p_i}(\sigma^n x)$;
- $\{A_n^i :\ 0 \leq n < p_i\}$ is a clopen partition of X;
- $A_m^j \subseteq A_n^i$ for $j > i$ and $n = m \mod p_i$;
- $\sigma(A_n^i) = A^i_{(n+1) \bmod p_i}$.

..., [R. Gjerde, R. Johansen, 2000]

Let W_1 be the collection of all words of length p_1, beginning with $x(0)$, we can make a Kakutani-Rokhlin tower $\mathcal{T}_1$ with columns based on the

$$B_w^1 := \{x \in A_0^1 : \ x[0, p_1 - 1] = w\}, \ w \in W_1.$$

So all the columns have the same heights p_1. Similarly, $\mathcal{T}_n$ will be a tower with columns bases

$$B_w^1 := \{x \in A_0^n : \ x[0, p_n - 1] = w\}, \ w \in W_n$$

which implies that all the columns have the height p_n. In other words,

$$\mathcal{T}_n = \{T^j B_w^1 : \ w \in W_n, \ j = 0, 1, \cdots, p_n - 1\}.$$

From CMS to a Bratteli diagram

Let (X, T) be a Cantor minimal system and consider a nested sequence of Kakutani-Rokhlin towers $\{\mathcal{T}_n\}_{n\geq 0}$ for that. We can realize this towers in the form of an infinite partially ordered graph such that

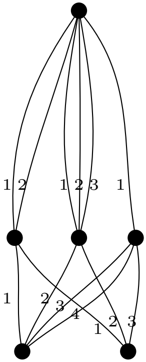
- at each level n associated to the tower $\mathcal{T}_n$, there are K_n vertices regarding the K_n columns of $\mathcal{T}_n$. The set of vertices of level n is denoted by V_n;
- for each two vertices in two consecutive levels, $u \in V_n$, $v \in V_{n+1}$, there are m edges connecting them regarding the m times of appearance of the column u as a sub-column of the column v;
- the edges terminated at each vertex in level $n+1$ are ordered and the ordering is related to the ordering of the columns of tower $\mathcal{T}_n$ as sub-columns of tower $\mathcal{T}_{n+1}$.

Bratteli diagram

A Bratteli diagram is a couple $B = (V, E)$ where

$$V = V_1 \dot{\sqcup} V_2 \dot{\sqcup} \cdots V_n \cdots , \quad E = E_1 \dot{\sqcup} E_2 \dot{\sqcup} \cdots E_n \cdots ,$$

and E_n is determined by an incidence matrix $|V_n| \times |V_{n-1}|$.

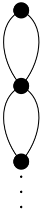


$$M(1) = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$M(2) = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

Example 1. odometer

As the Kakutani-Rokhlin towers have one columns at each level with $H_n = j_n j_{n-1} \cdots j_1$, the Bratteli diagram associated to the odometer $Z_{\mathbf{p}}$, $\mathbf{p} = (j_n)_{n\geq 1}$ have one vertex at each level with j_n edges between the vertices of two consecutive levels.



Example 2. substitutions

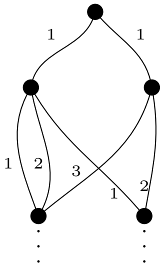
When $\tau : A \to A^+$, $A = \{a_1, a_2, \ldots, a_\ell\}$ is the substitution map, the Bratteli diagarm associated to (X_τ, σ) will have

- ℓ vertices at each level, $|V_n| = \ell$;
- For the number of edges between the levels, consider the incidence matrix associated to τ. Let M be an $\ell \times \ell$ matrix such that
  - M_{ij} shows that how many times the letter a_j appears in $\tau(a_i)$.
  - The ordering of the edges terminated at vertex $v_i \in V_1$ is the same as the order of letters in $\tau(a_i)$.
- Since *"the order of the appearance of the columns of each tower $\mathcal{T}_{n-1}$ as the sub-columns of the next tower $\mathcal{T}_n$, is the same as $\mathcal{T}_0$'s appear in $\mathcal{T}_1$,"* the Bratteli diagram associated to a substitution is *stationary* means that for all n, $M_n = M$.

The above construction was in fact based on the following theorem.

Theorem. (F. Durand, B. Host, C. Skau, 1999)

The family of substitution systems is in one to one correspondence with the family of stationary ordered Bratteli diagrams.

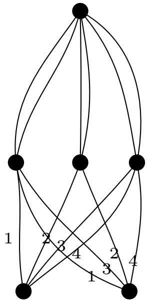


$$M(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$M(n) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Example 3. Toeplitz

The Bratteli diagram associated to a Toeplitz system is an ERS diagram, means that each incidence matrix have equal row sums. This is because of the heights of the columns of each Kakutani-Rokhlin tower which are all the same.



$$M(1) = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$M(2) = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$$

$$\vdots \qquad \qquad \vdots$$

From Bratteli diagram to CMS

- Vershik map: Let $(B, \leq)$ be an ordered Bratteli diagram and

$$x = (a_1, a_2, \cdots, a_{i_0}, \cdots)$$

  be an infinite path on it. Suppose that i_0 is the first i that a_i is not the max edge. Then

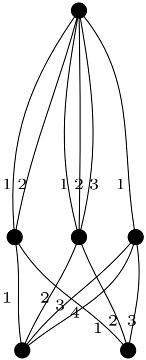
$$T(a_1, a_2, \cdots, a_{i_0}, \cdots) = (0, 0, \cdots, 0, \ a_{i_0} + 1, \cdots)$$
$$T(x_{\max}) = x_{\min}.$$

  So the map sends each infinite path to its successor.

- An Odometer:

$$\{0, 1, 2\}^{\mathbb{N}} \to \{0, 1, 2\}^{\mathbb{N}}$$

$$(2, 2, 2, 0, a, \cdots) \mapsto (0, 0, 0, 0 + 1, a, \cdots).$$



$$M(1) = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$M(2) = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

If the incidence matrices have all entries positive then the Vershik system is minimal.

Theorem. (T. Downarowicz, A. Maass, 2008)

Any Vershik system on a finite rank Bratteli diagram is conjugate to an odometer or to a subshift (expansive).

If the width of the diagram is infinite, this may not be true.

Theorem. (F. Sugisaki 2001)

A Vershik system on an ERS Bratteli diagram is strong orbit equivalent to a Toeplitz.

Gjerdeh and Johansen made example of a Vershik system on an ERS diagram which is neither subshift (expansive) nor an odometer.

Continuous spectrum and Bratteli diagram

Let (X, T) be a Cantor minimal system and consider the so called *Koopman operator, U_T,* on $C(X)$ defined by

$$U_T : C(X) \to C(X)$$
$$U_T(f) = f \circ T.$$

Definition.

A complex number $\lambda = \exp(2\pi it)$ is called an eigenvalue for (X, T) if it is an eigenvalue for the Koopman linear operator;

$$\exists\ f \in C(X); \quad U_T(f) = \lambda f.$$

Then the function $f : X \to \mathbb{R}$ is called an *eigenfunction*.

$$SP(T) := \{t;\ \exp(2\pi it) \text{ is eigenvalue for } (X, T)\} \neq \emptyset$$

is a countable additive subgroup of $\mathbb{R}$.

- Recall that the *measurable spectrum* for a dynamical system (X, T, μ) is defined similarly with Koopman operator on $L^2(\mu)$.
- The continuous spectrum is contained in the measurable spectrum.
- An invariant measure (even with full support) may have trivial continuous spectrum and non-trivial measurable spectrum.
- A (minimal) system is weakly mixing iff it has trivial (continuous) spectrum.

spectrum and Bratteli diagram

Let (X_B, T_B) be a Vershik map on an ordered Bratteli diagram.

Proposition. (Exercise)

The rational number $1/p$ *belongs to* $SP(T)$ *iff there exists some level* n *such that*

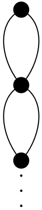
$$p | h_i, \quad 1 \leq i \leq |V_n|,$$

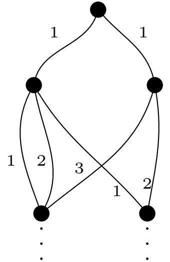
where h_i *is the number of paths from* $v_0 \in V_0$ *to* $v_i \in V_n$.

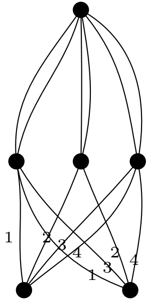
This means that the rational spectrum , $\mathbb{Q}(SP(T))$, is independent of the ordering of the Bratteli diagram.

The above proposition is indeed a corollary of [T, Giordano, I. Putnam, C. Skau, '95]

Examples.







For an ordered Bratteli diagram (X_B, T_B), having irrational spectrum is a non-invariant property under change of the ordering;

Proposition. (A direct corollary of Theorem 6.1, N. Ormes '95)

Let $(\hat{\mathbb{S}}^1, R_\theta, \ell)$ be the sturmian system with rotation number θ and invariant measure ℓ. Consider any (measure theoretically) weakly mixing system (Y, S, ν). There exists a system $(\hat{\mathbb{S}}^1, g)$ preserving λ and isomorphic to (Y, S, ν) such that $(\hat{\mathbb{S}}^1, R_\theta)$ and $(\hat{\mathbb{S}}^1, g)$ are realized as two different orderings on the same (telescoped) Bratteli diagram.

Proposition. (T. Giordano, D. Handelman, H., 2017)

Any Cantor minimal system with trivial **rational** *spectrum is strongly orbit equivalent to a weakly mixing system.*

Entropy and Bratteli diagram

Recall that for a subshift (X, σ), the entropy of σ is equal to

$$h(\sigma) = \limsup_n \frac{\log |\mathcal{W}_n(\sigma)|}{n},$$

where $\mathcal{W}_n(\sigma) = \{y_1 y_2 \dots y_n : \ \exists \ y = (y_i)_{i\in\mathbb{Z}} \in X\}$.

Note that any Vershik map T on an ordered Bratteli diagram $(B, V, \leq)$ is an inverse limit of subshifts:

$$T = \varprojlim_n (\sigma_k),$$

where σ_k is the subshift on the *quotient* of the space X_B obtained by restricting all the paths to the level k. Therefore,

$$h(T) = \lim_{k\to\infty} h(\sigma_k).$$

Proposition. (M. Boyle, D. Handelman, '94)

Let (X_B, T_B) be a Vershik system on $(B, V, \leq)$ which is consecutively ordered. Set n_k to be the minimum number of edges from a vertex at level $k-1$ to a vertex at level k and m_k be the number of vertices of level k. Suppose that

$$\lim_{k\to\infty} \frac{\log(n_k \cdot m_k)}{n_k} = 0.$$

Then the entropy of T_B is zero.

Corollary.

Any Cantor minimal system is strongly orbit equivalent to a system with zero entropy.

Proof. For any Bratteli diagram (B, V), there exists a relevant telescoping with the desired property of the proposition. Then any consecutive ordering will make the result.

Theorem. (M. Boyle, D. Handelman, '94)

Suppose $0 \leq \log \alpha \leq \infty$. *There exists a homeomorphism* T *strongly orbit equivalent to the odometer such that* $h(T) = \log \alpha$.

Theorem. (Downarowicz, Lacorix, 1998)

Let (X, T, μ) *be an ergodic system with countably many rational (measurable) spectrum. There exists a uniquely ergodic Toeplitz system* (X, T) *with an invariant measure* ν *which is measure theoretically isomorphic to* (X, T, μ).

Theorem. (Siri Malen, 2015)

For any $0 \leq t \leq \infty$*, any Choqute simplex* K *and any odometer* $Z_{\mathbf{p}}$*, there exists Toeplitz flow* (X, T) *with entropy equal to* t*, maximal equicontinuous factor* $Z_{\mathbf{p}}$ *and with the set of invariant measures affinely homeomorphic to* K*.*

Some References.

- M. Boyle, *Topological Orbit Equivalence and Factor Maps in Symbolic Dynamics*, Ph.D. Thesis, University of Washington, Seattle (1983).
- M. Boyle, D. Handelman, Entropy Versus Orbit Equivalence For Minimal Homeomorphisms, *Pasific J. Math.*, Vol 164, N0. 1, (1994), 1-13.
- Dekking, F. M. and Keane, M., Mixing Properties of Substitutions, *Z. Wahrschein- lichkeitstheorie verw. Gebiete,* 42 (1978), 23-33.
- T. Downarowicz, Y. Lacroix, Almost One to One extensions of Furstenberg-Wiess type and Applications to Toeplitz Systems, *Studia Mathematica*, **130** (2) (1998), 149-170.

- T. Downarowicz, A. Maass, Finite Rank Bratteli- Vershik Diagarams Are Expansive, *Erg. Th. and Dyn. Sys.*, Vol 28, Issue 3, (2008), 739-747.
- F. Durand, B. Host, C. Skau, Substitutional dynamical systems, Bratteli diagrams and Dimension groups, *Ergodic Th. and Dyn. Dyd.* **19** (1999), 953-993.
- E. Glasner, B. Wiess, Weak Orbit Equivalence of Cantor Minimal systems, *Intern. J. Math*, **6** (1995), 569-579.
- T. Giordano, D. Handelman, M. Hosseini, Orbit Equivalence of Cantor Minimal Systems and their Continuous Spectra, http://arxiv.org/abs/1606.03824.
- T. Giordano, I. Putnuam, and C. Skau, Topological Orbit Equivalence and C*-crossed Product, *J. Reine Angew. Math.*, **469** (1995), 51-111.

- R. Gjerde, ϕ. Johnsen, Bratteli-Vershik Models for Cantor Minimal Systems: Applications to Toeplitz Flows, *Ergod. Th. and Dynam. Sys.*, **20** (2000), 1687-1710.
- E. Glasner, B. Wiess, Weak Orbit Equivalence of Cantor Minimal systems, *Intern. J. Math*, **6** (1995), 569-579.
- R. H. Herman, I. F. Putnuam and C. F. Skau, Ordered bratteli diagrams, Dimension groups and topological dynamics, *Int. J. Math.*, **3** (1992), 827-864.
- S. M. Hoynes, Toeplitz Flows and Their Ordered K-theory, *Ergodic Theory and Dynamical Systems*, *Ergodic Th. and Dyn. Sys.*, Vol 36, Issue 6, (2016), 1892-1921.

- P. Kurka, *Topological and symbolic dynamics*, Societe Mathematique de France, 2003.
- N. Ormes, Strong Orbit Realization for Minimal Homeomorphisms, *J. Anal. Math.*, **71** (1997), 103-133.
- I. F. Putnam, The C*-Algebras Associated with Minimal Homeomorphisms of the Cantor Set, *Pasific J. Math.*, Vol. 136, No. 2, (1989), 329-353.
- M. Queffelec, Substitution Dynamical Systems (Lecture Notes in Mathematics, 1294). Springer, 1987.
- F. Sugisaki, Toeplitz Flows, Orederd Bratteli Diagrams and Strong Orbit Equivalence, *Ergodic Theory and Dynamical Systems*, **21** 06, (2001), 1867-1881.