Symbolic Dynamics

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Outline

1. Basic definitions and applications of general subshifts,
2. Coded systems, in particular subshifts of finite type and sofics.
3. A brief introducing spacing shifts as a source for examples which are neither coded nor minimal,
4. Interaction between, shifts, topological dynamics and ergodic theory,
5. Some major problems in symbolic dynamics.

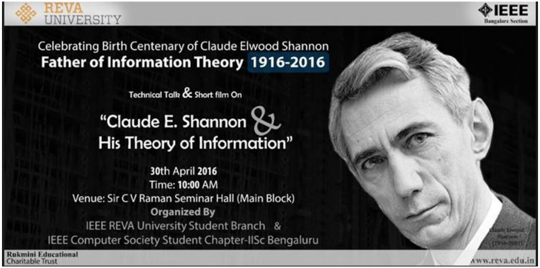
History

Jacques Salomon Hadamard
Born 8 December 1865 (France)
Died 17 October 1963 (aged 97)

- J. Hadamard (1898), "Les surfaces à courbures opposés et leurs lignes géodésiques" . J. Math. Pures Appl. 5 (4): 27-73.
- M. Morse and G. A. Hedlund (1938), "Symbolic Dynamics". American Journal of Mathematics, 60: 815-866.
- George Birkhoff, Norman Levinson and the pair Mary Cartwright and J. E. Littlewood use it for nonautonomous second order differential equations.

History

- Claude Shannon used symbolic sequences and shifts of finite type in his 1948 paper "A mathematical theory of communication".



- Smale gives the global theory of dynamical systems in 1967.
- Roy Adler and , Benjamin Weiss applied them to hyperbolic toral automorphisms.
- Yakov Sinai used them in Anosov diffeomorphisms.
- In the early 1970s the theory was extended to Anosov flows by Marina Ratner, and to Axiom A diffeomorphisms and flows by Rufus Bowen.

History

Marina Ratner

Born January 10, 1938 (Russia)
Professor at the University of California, Berkeley (age 77)

Rufus Bowen

Born 23 February 1947
Died 30 July 1978 (aged 31)

History

Now symbolic dynamics is applied in many areas within dynamical systems such as

- Maps of the interval.
- Billiards.
- Complex dynamics.
- Hyperbolic and partially hyperbolic diffeomorphisms and flows.

And outside in

- Information theory.
- Matrix theory
- Automata theory.

Preliminaries

Let $\mathcal{A} \cong \{0,\, 1,\, \cdots, k-1\}$ be a set of k characters called **alphabet**.

- $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ is called the **full shift** where σ is the shift map:

$$x = \cdots x_{-2}x_{-1\wedge}x_0x_1x_2\cdots \mapsto \sigma(x) = \cdots x_{-2}x_{-1}x_{0\wedge}x_1x_2\cdots\,; \text{ or}$$

$$\sigma(x)_i = x_{i+1}.$$

- A finite sequence of characters $x_0x_1\cdots x_{k-1}$ is called a **block** or **word**.
- A **shift space** or **subshift** (or simply **shift**) is (X, σ_X) where X is a subset of the full shift such that $X = X_{\mathcal{F}}$ for some collection $\mathcal{F}$ of forbidden blocks over $\mathcal{A}$ and $\sigma_X = \sigma|_X$. When no ambiguity arises we denote σ_X with σ. Note that full shift is $X_{\mathcal{F}}$ with $\mathcal{F} = \emptyset$.

Example

If $\mathcal{A} = \{0,\, 1\}$ and $\mathcal{F} = \{11\}$, then $X_{\mathcal{F}}$ is called the **golden mean shift**.

Let X be a subshift. Then,

- $\mathcal{B}_n = \{x_0 x_1 \cdots x_{n-1} :\ \exists x \in X \text{ s.t. } x = (x_i)_i = \cdots x_{-2} x_{-1} x_0 x_1 \cdots x_{n-1} x_n \cdots\}$.
- $\mathcal{B}(X) = \cup_{n=0}^{\infty} \mathcal{B}_n$ is called the **language** of X.
- The language determines the subshift. Because, $X = X_{\mathcal{F}}$ where $\mathcal{F} = \mathcal{B}(X)^c$.

An **(invertible) dynamical system** is a set X , together with an (invertible) mapping $T : X \to X$.
Let $x \in X$

$$\underbrace{\overbrace{\cdots, T^{-2}(x), T^{-1}(x),}^{\text{backward orbit}} \overbrace{x, T(x), T^2(x) = T \circ T(x), \cdots}^{\text{forward orbit}}}_{\text{orbit}}.$$

Generalizations

Let Γ be a countable semigroup and consider X and the action of Γ on X as follows

$$X = \{\sigma : \ \sigma : \Gamma \to \mathcal{A}\}, \quad \Gamma \times X \to X, \quad \gamma\sigma(\gamma') = \sigma(\gamma'\gamma).$$

Equip Γ and $\mathcal{A}$ with discrete topology and X with the product topology. (X, σ) is called the **Bernouli shift** associated to Γ and $\mathcal{A}$. By this topology X is a Cantor set.

Exercises:

1. Give a subset Z of the full shift such that (Z, σ_Z) is *not* a subshift.
2. Assume that for $i = 1,\ 2$; (X_i, σ_{X_i}) is a subshift. Show that $(X_1 \times X_2, \sigma_{X_1 \times X_2})$ is also a subshift.
3. Is the union of finitely many subshifts a subshift? What about the intersection?
4. Show that any full shift has uncountably many points and give an example of $\{0, 1\}^{\mathbb{Z}}$ with only infinitely *countable* points.
5. Show that if $|\mathcal{F}| < \infty$, then periodic points of $X = X_{\mathcal{F}}$ are dense in X and X is *transitive*; that is, there is a point $x \in X$ so that
$$X = \overline{\{\sigma_X^n(x) :\ n \in \mathbb{Z}\}}.$$
6. Give an example of a subshift X, $|X| = \infty$ and with only finitely many periodic points.
7. If $Y \subseteq X$ and (Y, σ_Y) is a subshift, then (Y, σ_Y) is called a **subsystem** of (X, σ_X). Show that full shift on $\{0, 1\}$ has uncountably many subsystems.
8. Give an example of a subshift with exactly two subsystems.

Sushifts as metric spaces

The full shift is endowed with the product topology on $\mathcal{A}^{\mathbb{Z}}$. By this topology σ and σ^{-1} are continuous. The metric on X is

$$d(x, x') = \begin{cases} 0, & \text{if } x = x'; \\ 2^{-k}, & \text{if } x \neq x',\, k = \max_i \; x_{-i} \dots x_i = x'_{-i} \dots x'_i. \end{cases}$$

Excercises:

1. Any subshift is a closed subspace of $\mathcal{A}^{\mathbb{Z}}$.
2. ${}_k[a_k \dots a_\ell]_\ell = \{(x_i)_i \in X : \; x_k = a_k, \dots, x_\ell = a_\ell\}$ called a **cylinder** is a component of X.
3. Show that the set of all cylinders in a subshift X is a basis for a topology equivalent to the topology of the aforementioned metric on X. Notice that by this topology, any subshift is Hausdroff and satisfies second axiom of countability.
4. Show that by the above metric, any subshift is bounded and find its diameter.
5. Let $x = \dots x_{-1}x_0x_1 \dots$ and set $y_i = x_{-i}$. Then show that $h : \mathcal{A}^{\mathbb{Z}} \to [0,\, 1] \times [0,\, 1]$ defined as $x \mapsto (0.x_0x_1 \dots,\, 0.y_1y_2 \dots)$ is continuous.

Examples for one sided shifts

If $x \to 2x \mod 1$ on $[0, 1]$ and the partition is $\{[0, 1/2], [1/2, 1]\}$, then one obtains all one-sided binary sequences; If $x \to \lambda x \mod 1$ where $\lambda = \frac{1+\sqrt{5}}{2} = 1.61803\ldots$ is the golden ratio and the partition is $\{[0, 1/\lambda], [1/\lambda, 1]\}$, then one obtains all one-sided sequences that do not contain two consecutive 1's.

Definition

A shift space X is **irreducible** if for every ordered pair of blocks u, v in $\mathcal{B}(X)$, there is a w so that $uwv \in \mathcal{B}(X)$. X is **mixing**, if for every ordered pair of blocks u, v there is $M = M(u, v) \in \mathbb{N}$ such that for any $n \geq M$ there is $w \in \mathcal{B}_n(X)$ with $uwv \in \mathcal{B}(X)$.

Exercise:

Is the golden mean shift irreducible? mixing?!

Applications

Let (M, ϕ) be an "invertible" dynamical system. $\mathcal{P} = \{P_0, \cdots, P_r\}$ a topological partition on M and $\mathcal{A} = \{0, \ldots, r\}$. $w = a_1 a_2 \ldots a_n$ is **admissible** word for $\mathcal{P}$ and ϕ if $\cap_{j=1}^{n} \phi^{-1}(P_{a_j}) \neq \emptyset$.

- The collection of all admissible words is a language for a shift space $X = X(\mathcal{P}, \phi)$.
- For $x = (x_i)_i \in X$ and $n \geq 0$ set $D_n(x) = \cap_{i=-n}^{n} \phi^{-i}(P_{x_i})$. Then $D_n(x)$ is open and $\overline{D_0} \supseteq \overline{D_1} \supseteq \overline{D_2} \supseteq \cdots$. Clearly $\cap_{n=0}^{\infty} \overline{D_n}(x) \neq \emptyset$.

Definition

Topological partition of M gives a **symbolic representation** for (M, ϕ) if $x \in X \Rightarrow |\cap_{n=0}^{\infty} \overline{D_n}(x)| = 1$.

- (X, σ) is transitive iff (M, ϕ) is,
- (X, σ) is mixing iff (M, ϕ) is,
- (X, σ) has a set of dense periodic set iff (M, ϕ) does.

Some of the Applications

- Markov partitions for hyperbolic toral automorphisms give rise to shifts of finite type.
- Sinai used the symbolic dynamics in the study of Anosov diffeomorphisms and (Smale, Bowen, Manning, etc.) for Axiom A diffeomorphism. For instance, they proved that the ζ-function for an Axiom A diffeomorphism is a rational function.

Factors and Codes

Let (X, T) and (Y, S) be topological dynamical systems. Then $\varphi : X \to Y$ is called a **homomorphism** if

- φ is continuous, and
- $\varphi \circ T = S \circ \varphi$.

if φ is onto, then it is called **factor**; and if φ is homeomorhpism, then it is called **conjugacy**.

Let m, n be integers with $-m \leq n$ and set $\ell = m + n = 1$. Let $\mathcal{A}'$ be another set of alphabet. $\Phi : \mathcal{B}_\ell \to \mathcal{A}'$ is called an ℓ-**block map**.

Definition

The map $\phi : X \to X' \subseteq \mathcal{A}'^{\mathbb{Z}}$; $x = (x_i)_i \mapsto \phi(x) = (x'_i)_i$ by

$$x'_i = \Phi(x_{i-m}x_{i-m+1} \cdots x_{i-1}x_ix_{i+1} \cdots x_{i+n}),$$

is called the **sliding block code** with **memory** m and **anticipation** n induced by Φ. It will be denoted by $\phi = \Phi_\infty^{[-m,n]}$, or just $\phi = \Phi_\infty$ when m and n are understood.

$$\begin{array}{ll} x & = \cdots x_{i-m-1}x_{i-m}x_{i-m+1} \cdots x_{i-1}x_i\,x_{i+1} \cdots x_{i+n}x_{i+n+1} \cdots \\ \phi\downarrow & \qquad\qquad\qquad\qquad\qquad \Phi\downarrow \\ x' & = \cdots x_{i-m-1}x'_{i-m}x'_{i-m+1} \cdots x'_{i-1}x'_i\,x_{i+1} \cdots x'_{i+n}x'_{i+n+1} \cdots \end{array}$$

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Exercises:

1. Show that $\phi : X \to X'$ is a factor map iff it is onto and is a sliding block code. Hence ϕ as a sliding block code is continuous and the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\sigma_X} & X \\ \phi \big\downarrow & & \phi \big\downarrow \\ X' & \xrightarrow{\sigma_{X'}} & X' \end{array}$$

commutes.

2. A sliding block code ϕ is a conjugacy iff ϕ^{-1} is a sliding block code.
3. A sliding block code ϕ is a conjugacy iff it is onto and 1-1.
4. A sliding block code preserves both irreducibility and mixing.
5. Give an example of a sliding block code ϕ s.t. X' is irreducible (resp. mixing), but X not being irreducible (resp. mixing).

Shifts of Finite Type (SFT)

Definition

Assume there is $\mathcal{F} \subset \mathcal{A}^{\mathbb{Z}}$ such that $|\mathcal{F}| < \infty$ and $X = X_{\mathcal{F}}$. Then, X is called the **shift of finite type** or SFT.

Exercise:
Show that there is $X = X_{\mathcal{F}_1} = X_{\mathcal{F}_2}$ such that $|\mathcal{F}_1| < \infty$ and $|\mathcal{F}_2| = \infty$.

Example

- Golden mean shift and all full shifts are SFT.
- **A non SFT example**. Let $\mathcal{F} = \{10^{2n+1}1 : \ n \in \mathbb{N} \cup \{0\}\}$. Then $X_{\mathcal{F}}$ is called the **even shift** and it is **not** SFT. Otherwise, there is a finite $\mathcal{F}' \subset B(X_{\mathcal{F}})^c$ s.t. $X_{\mathcal{F}} = X_{\mathcal{F}'}$ and if $u \in \mathcal{F}'$ then $|u| = N \in \mathbb{N}$. Now $0^\infty 10^{2N+1}10^\infty \in X_{\mathcal{F}}$ which is absurd.

Shifts of Finite Type (SFT), Cont...

Definition

Assume $X = X_{\mathcal{F}}$ where $\mathcal{F}$ is finite and for $u \in \mathcal{F}$ we have $|u| = M + 1$. Then X is called an M-**step** SFT.

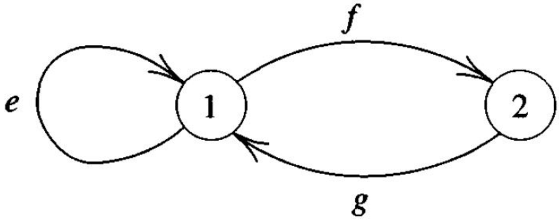
- X is a 0-step SFT iff X is a full shift.
- If X is an M-step SFT, then it is K-step SFT for $K \geq M$.
- If X is an SFT, then there is an $M \geq 0$ such that X is M-step.

Theorem

$X = X_{\mathcal{F}}$ is an M-step SFT iff whenever uv, vw are admissible (not in $\mathcal{F}$) and $|v| \geq M$, then uvw is admissible as well.

Graph

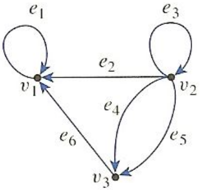
This is a (directed) graph!!



- $\mathcal{V} = \{①, ②\}$ is the set of vertices or states.
- $\{e, f, g\}$ are the labels of edges in $\mathcal{E}$.
- $i(f) = i(e) = t(g) = ①$ and $t(f) = i(g) = ②$.
- An edge e with $i(e) = t(e)$ is called a **self-loop**.
- A graph homomorphism can be defined naturally between two graphs G and H.

Adjacency Matrix

Let G be a (directed) graph with vertex set $\mathcal{V} = \{v_1, v_2, \ldots\}$. Let A_{ij} denote the number of edges in G from $v_i \in \mathcal{V}$ to $v_j \in \mathcal{V}$. Then the **adjacency matrix** of G is $A = A_G = [A_{ij}]$.



Directed Graph G

$$\mathbf{A} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

Adjacency Matrix

- Here we only consider cases where $|\mathcal{V}| < \infty$.
- $A^k = [A^k_{ij}]$ where A^k_{ij} is the number of "paths" of length k from $v_i \in \mathcal{V}$ to $v_j \in \mathcal{V}$.

Definition

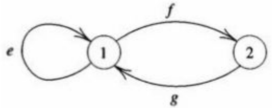
Let G be a graph and A its adjacency matrix. The **edge shift** X_G (or X_A) is the shift space over the alphabet $\mathcal{A} = \mathcal{E}$ defined as

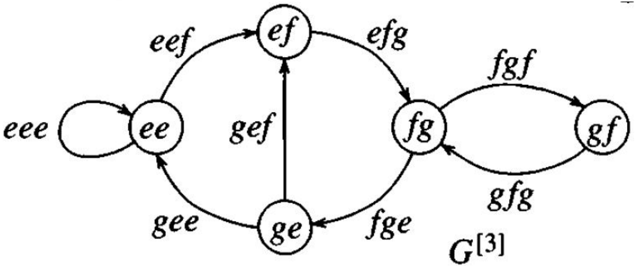
$$X_G = X_A = \{\xi = (\xi_i)_i \in \mathcal{A}^{\mathbb{Z}} : \forall i,\ t(\xi_i) = i(\xi_{i+1})\}.$$

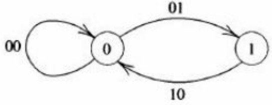
The **edge shift map** is the shift map defined on X_G (or X_A) and is denoted by σ_G (or σ_A).

- Any point in X_G describes a bi-infinite walk on G.
- Here we assume $\mathcal{V}(G)$ is finite. Then, X_G is a 1-step SFT.
- trace(A^p) is the number of cycles of length p in G which in turn equals the number of points in X_G with period p.
- All the examples of graphs given above are **irreducible** or **strongly connected**.

Nth Higher Edge Graph $G^{[N]}$







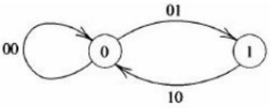
Definition

Let B be the adjacency matrix of a graph G with r vertices such that between any two vertices there is at most one edge. The **vertex shift** $\hat{X}_B = \hat{X}_G$ is the shift space with alphabet $\mathcal{A} = \{1, 2, ..., r\}$, defined by

$$\hat{X}_B = \{(x_i)_i \in \mathcal{A}^{\mathbb{Z}} : \ B_{x_i x_{i+1}} = 1 \ \forall i \in \mathbb{Z}\}.$$

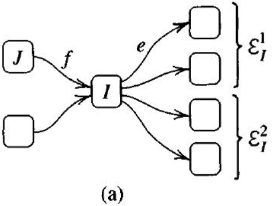
The **vertex shift map** is the shift map on $\hat{X}_B$ and is denoted by $\hat{\sigma}_B$.

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

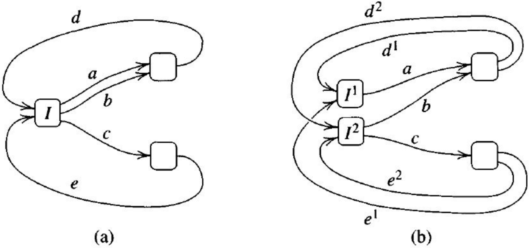


Theorem

If X is an M-step SFT, then $X^{[M]}$ is a 1-step SFT, equivalently a vertex shift. In fact, there is a graph G s.t. $X^{[M]} = \hat{X}_G$ and $X^{[M+1]} = X_G$.

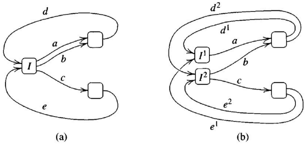


Splitting a single state.



Elementary state splitting of $\mathcal{E}_I = \{a, b, c\}$ to $\mathcal{E}_I^1 = \{a\}$ and $\mathcal{E}_I^2 = \{b, c\}$. (Here, $\mathcal{E}^I = \{d, e\}$.)

Conjugacy by State Splitting



Let X_G (resp. X_H) be the edge shift associated to graph (a) (resp. (b)). Define Ψ to be the 1-block map with

$$\Psi(f^i) = f, \text{ if } f \in \mathcal{E}^I \quad \text{and} \quad \Psi(e) = e \text{ if } e \notin \mathcal{E}^I.$$

Let $\psi = \Psi_\infty : X_H \to X_G$. Let $\Phi : \mathcal{B}_2(X_G) \to \mathcal{B}_1(X_H)$ by

$$\Phi(fe) = \begin{cases} f & \text{if } f \notin \mathcal{E}^I, \\ f^1 & \text{if } f \in \mathcal{E}^I \text{ and } e \in \mathcal{E}_I^1, \\ f^2 & \text{if } f \in \mathcal{E}^I \text{ and } e \in \mathcal{E}_I^2. \end{cases}$$

Set $\phi = \Phi_\infty : X_G \to X_H$. Then

$$\psi(\phi(x)) = x, \qquad \text{and} \qquad \phi(\psi(y)) = y.$$

Thus $\phi : X_G \to X_H$ is a conjugacy. Above example was an

out-splitting, **in-splitting** can be done similarly. If a graph H is a splitting of a graph G, then G is called an **amalgamation** of H.

Theorem

If a graph H is a splitting of a graph G, then the edge shifts X_G and X_H are conjugate.

Decomposition Theorem

Theorem

Every conjugacy from one edge shift to another is the composition of splitting codes and amalgamation codes.

In other words, let G and H be graphs. The edge shifts X_G and X_H are conjugate iff G is obtained from H by a sequence of out-splittings, in-splittings, out-amalgamations, and in-amalgamations.

State Split Graph

Let $\mathcal{P}$ be a partition in $\mathcal{E}(X)$.

$$\mathcal{P} = \bigcup_{I \in \mathcal{V}} \mathcal{P}_I, \text{ where } \mathcal{P}_I = \{\mathcal{E}_I^1, \ldots, \mathcal{E}_I^{m(I)}\}.$$

$H = G^{[\mathcal{P}]}$ is called **out-state split graph** with

$$\mathcal{V}(H) = \bigcup_{I \in \mathcal{V}(G)} \{I^1, \ldots, I^{m(I)}\},$$

$$\mathcal{E}(H) = \left\{ e^j : \ e \in \mathcal{E}_I^i, \, 1 \leq j \leq m(t(e)), \ (I \xrightarrow[G]{e} J \Rightarrow I^i \xrightarrow[H]{e^j} J^j) \right\}.$$

Let G and $H = G^{[\mathcal{P}]}$ be as above.

Definition

The **division** matrix for $\mathcal{P}$ is the $|\mathcal{V}(G)| \times |\mathcal{V}(H)|$ defined as

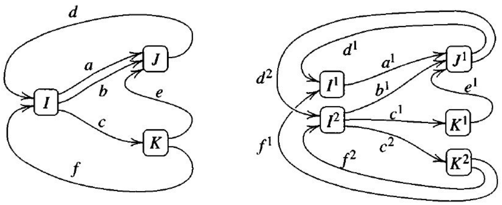
$$D(I,\, J^k) = \begin{cases} 1 & \text{if } I = J, \\ 0 & \text{otherwise.} \end{cases}$$

The **edge** matrix for $\mathcal{P}$ is the $|\mathcal{V}(H)| \times |\mathcal{V}(G)|$:

$$E(I^k,\, J) = |\mathcal{E}_I^k \cap \mathcal{E}^J|.$$

Theorem

$$DE = A_G \quad \textit{and} \quad ED = A_H.$$



(a) (b)

$$\mathcal{P} = \left\{\mathcal{E}_I^1 = \{a\},\, \mathcal{E}_I^2 = \{b,\, c\},\, \mathcal{E}_J^1 = \{d\},\, \mathcal{E}_K^1 = \{e\},\, \mathcal{E}_k^2 = \{f\}\right\}.$$

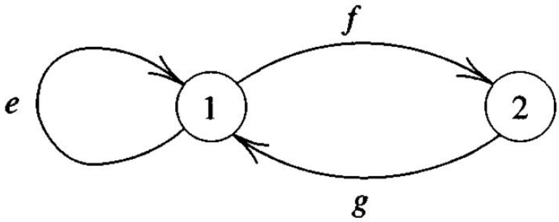
$$D = \begin{array}{c} \\ I \\ J \\ K \end{array} \begin{array}{c} \begin{array}{ccccc} I^1 & I^2 & J^1 & K^1 & K^2 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

$$E = \begin{array}{c} \\ I^1 \\ I^2 \\ J^1 \\ K^1 \\ K^2 \end{array} \begin{array}{c} \begin{array}{ccc} I & J & K \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

Labeled Graph

Definition (labeled graph)

A **labeled graph** S is a pair $(G, \mathcal{L})$, where G is a graph with **edge set** $\mathcal{E}$, and the labeling $\mathcal{L} : \mathcal{E} \to \mathcal{A}$ assigns to each edge e of G a label $\mathcal{L}(e)$. The **underlying graph** of S is G. A labeled graph is **irreducible** if its underlying graph is irreducible.



Sofics

There are several equivalent definitions for a shift of sofic.

Definition

A factor of an SFT is called **sofic**.

Let $\xi = \dots e_{-1}e_0e_1 \dots \in X_G$ be an infinite walk in G. **Label of the walk** is

$$\mathcal{L}(\xi) = \dots \mathcal{L}(e_{-1})\mathcal{L}(e_0)\mathcal{L}(e_1) \dots \in \mathcal{A}^{\mathbb{Z}}.$$

Let

$$X_{\mathcal{G}} = \{\mathcal{L}_\infty(\xi) : \xi \in X_G\} = \mathcal{L}_\infty(X_G) \subseteq \mathcal{A}^{\mathbb{Z}}.$$

Theorem

A subset X of $\mathcal{A}^{\mathbb{Z}}$ is a sofic shift iff $X = X_{\mathcal{G}}$ for some labeled graph $\mathcal{G}$.

Definition

A **presentation** or **cover** of a sofic shift X is a labeled graph $\mathcal{G}$ for which $X = X_{\mathcal{G}}$.

Sofics...

- Cover for a sofic is not unique.
- Any SFT is a sofic. In fact,

  full shifts ⊆SFT's⊆ sofics ⊆ shift spaces.
- A non-SFT sofic is called **strictly sofic**. Even shift is such an example.
- A sofic shift is an SFT iff it has a cover $(G, \mathcal{L})$ such that $\mathcal{L}_\infty$ is a conjugacy.
- X is sofic iff it has a finite cover.
- Sofics are similar to regular languages in automata theory.
- The smallest collection of shift spaces that includes the shifts of finite type and is invariant under factor codes are sofic.

Definition

Let X be a subshift and $w \in \mathcal{B}(X)$. Then

$$F_X(w) = \{v \in \mathcal{B}(X) : \ wv \in \mathcal{B}(X)\}.$$

is called the **follower set** of $w \in X$. Set $\mathcal{C}_X := \{F_X(w) : \ w \in \mathcal{B}(X)\}$.

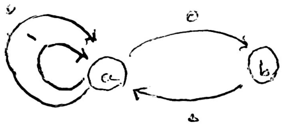
Assume X is a space with $\mathcal{C}_X$ finite. The **follower set graph** is a labeled graph $\mathcal{G} = (G, \mathcal{L})$ with

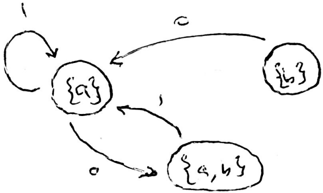
$$\mathcal{V}(G) = \mathcal{C}_X,$$
$$\mathcal{E}(G) = \{e =\!\rightarrow: \ F_X(w) \xrightarrow[G]{a} F_X(wa)\}.$$

Such an X must be sofic. In fact, a subshift is sofic if and only if it has a finite number of follower sets.

Existence of right resolving cover

A labeled graph $\mathcal{G} = (G, \mathcal{L})$ is **right resolving** if for $I \in \mathcal{V}(G)$, the edges starting at I carry different labels. Every sofic shift has a right-resolving presentation.





Definition

Among all right-resolving presentations of a sofic X, there is a right-resolving presentation of X having the least vertices called a **minimal right-resolving presentation** or **Fischer cover**.

Definition

Let $\mathcal{G} = (G, \mathcal{L})$ be a labeled graph, and $I \in \mathcal{V}$. Then

$$F_{\mathcal{G}}(I) = \{\mathcal{L}(\pi) : \pi \in \mathcal{B}(X_G) \text{and } i(\pi) = I\}.$$

is called the **follower set of I in $\mathcal{G}$**. $\mathcal{G}$ is called **follower-separated** if $I \neq J \Rightarrow F_{\mathcal{G}}(I) \neq F_{\mathcal{G}}(J)$.

Right-resolving Representation

Theorem

Sofics admit a Fischer cover, or equivalently a minimal right-resolving presentation.

- Sofic shifts have interesting, describable dynamical behavior (e.g., explicitly computable entropy and zeta functions).
- Sofic shifts can be described in a concrete and simple manner via labeled graphs.

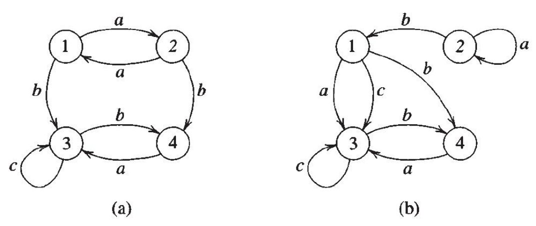
Question:

How many Fischer covers a sofic shift possesses?

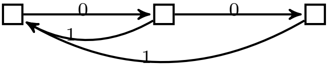
Examples

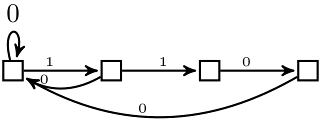
Example (Jonoska (1996))

An example of a sofic shift with two Fischer covers



Other examples:





Fischer Cover

Theorem (Fischer)

Up to isomorphism of labeled graphs, a Fischer cover of an "irreducible" sofic is unique.

Corollary

Let X be an irreducible sofic shift. Then a right-resolving graph $\mathcal{G}$ is a Fischer cover of X iff it is irreducible and follower-separated.

Definition

Synchronizing Word In a a labeled graph $\mathcal{G} = (G, \mathcal{L})$, a word $w \in \mathcal{B}(X_{\mathcal{G}})$ is a **synchronizing word** for $\mathcal{G}$ if all paths in G labeled as w terminate at the same vertex. If this vertex is I, we say that w **focuses** to I. A word w in a subshift X is synchronizing, if for uw and wv are words in X, then uwv is also a word in X. A system with a synchronizing word is called **synchronized** system.

- In a sofic system, the follower set of a synchronizing word equals the follower set of its focusing vertex in its Fischer cover.
- Any prolongation of a synchronizing word is synchronizing.
- Any sofic is synchronized. Hence,

$$\text{full shifts} \subset \text{SFT's} \subset \text{sofics} \subset \text{synchronized systems} \subset \text{subshifts}$$

- If $\varphi : X \to Y$ is a factor code and X synchronized, then Y is **not** necessarily synchronized.
- Let X be sofic and $\mathcal{G}_X = (G_X, \mathcal{L}_X)$ its Fischer cover, then $\mathcal{L}_\infty(\mathcal{G}_X) = X$.
- X is sofic iff there is a labeled graph $\mathcal{G}_X = (G_X, \mathcal{L}_X)$ s.t. $\mathcal{L}_\infty(\mathcal{G}_X) = X$.
- X is synchronized iff there is a labeled graph $\mathcal{G}_X = (G_X, \mathcal{L}_X)$ s.t. $\mathcal{L}_\infty(\mathcal{G}_X)$ is residual in X.
- If X is an irreducible synchronized, then there is a unique (up to isomorphism) right resolving and follower separated presentation for X, called the bf Fischer cover.
- There are few sofic shifts!

Entropy and its Properties

Definition

The **entropy** of a subshift X is defined by

$$h(X) = \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{B}_n(X)|.$$

- The above limit exists and equals $\inf_{n\in\mathbb{N}} \frac{1}{n} \log |\mathcal{B}_n(X)|$.
- $|\mathcal{B}_n(X)| \leq |\mathcal{A}|^n \Rightarrow h(X) \leq \log |\mathcal{A}|$.
- Let X be a full k-shift. Then, $h(X) = \log k$.
- For an SFT X_A, $|\mathcal{B}_n(X)| = \Sigma_{I,J=1}^{r} (A^n)_{IJ}$ where $r = |\mathcal{V}(G_A)|$.
- If Y is a factor of X, then $h(Y) \leq h(X)$. So conjugacy preserves entropy.
- If Y is a subsystem of X, then $h(Y) \leq h(X)$.

Entropy and ...

- $h(X^k) = kh(X)$, $h(X \times Y) = h(X) + h(Y)$ and $h(X \cup Y) = \max\{h(X), h(Y)\}$.
- Let $\mathcal{G} = (G, \mathcal{L})$ be a right-resolving labeled cover for the sofic X. Then, $h(X_{\mathcal{G}}) = h(X_G)$.
- Let $p_n(X)$ denote the number of points in X with period n. Then,

$$\limsup_{n \to \infty} \frac{1}{n} \log p_n(X) \leq h(X).$$

Entropy for an SFT

Theorem (Perron-Frobenius)

Let $A \neq 0$ be an irreducible matrix. Then A has a positive eigenvector v_A with corresponding eigenvalue λ_A that is both geometrically and algebraically simple. If λ_i is another eigenvalue for A, then $|\lambda_i| \leq \lambda_A$. Any positive eigenvector for A is a positive multiple of v_A.

Consider X_A and let $v_A = \{v_1, \ldots, v_r\}$. Let $c = \min_i v_i$ and $d = \max_i v_i$.Then,

$$c\sum_{J=1}^{r}(A^n)_{IJ} \leq \sum_{J=1}^{r}(A^n)_{IJ}v_J = \lambda^n v_I \leq d\lambda^n \Rightarrow \left(\sum_{I,J=1}^{r}(A^n)_{IJ} \leq \frac{rd}{c}\lambda^n\right).$$

Similarly, $(\frac{rc}{d})\lambda^n \leq \sum_{I,J=1}^{r}(A^n)_{IJ}$. So

$$h(X_A) = \log \lambda_A.$$

Let $p_n(X)$ be as before and $q_n(X)$ the number of periodic points of least period n.

Theorem

If X is an irreducible sofic shift, then

$$\limsup_{n\to\infty} \frac{1}{n} \log p_n(X) = \limsup_{n\to\infty} \frac{1}{n} \log q_n(X) = h(X).$$

Zeta Function

Definition

Let (M, ϕ) be a dynamical system with $p_n(\phi)$ finite for all $n \in \mathbb{N}$. Then zeta function $\zeta_\phi(t)$ is defined as

$$\zeta_\phi(t) = \exp\left(\sum_{n=1}^{\infty} \frac{p_n(\phi)}{n} t^n\right).$$

Assume $\zeta_\phi(t)$ has a positive radius of convergence. Then $\log \zeta_\phi(t) = \sum_{n=1}^{\infty} \frac{p_n(\phi)}{n} t^n$; and so

$$\frac{\frac{d^n}{dt^n} \log \zeta_\phi(t)|_{t=0}}{n!} = \frac{p_n(\phi)}{n}.$$

Zeta Function for an SFT

Let X_A be an SFT where A is an $r \times r$ square matrix. Then,

$$p_n(\sigma_A) = \operatorname{tr} A^n = \sum_{i=1}^{r} \lambda_i^n.$$

Thus,

$$\zeta_{\sigma_A}(t) = \exp\left(\sum_{n=1}^{\infty} \frac{(\lambda_1 t)^n}{n} + \cdots + \sum_{n=1}^{\infty} \frac{(\lambda_r t)^n}{n}\right).$$

By the fact that $-\log(1-t) = \sum_{n=1}^{\infty} \frac{t^n}{n}$, one has

$$\zeta_{\sigma_A}(t) = \frac{1}{1-\lambda_1 t} \times \cdots \times \frac{1}{1-\lambda_r t}.$$

Also, $\chi_A(\xi) = (\xi - \lambda_1)\cdots(\xi - \lambda_r) = \xi^r \det(Id - \xi^{-1}A)$. So

Theorem

$$\zeta_{\sigma_A}(t) = \frac{1}{t^r \chi_A(t^{-1})} = \frac{1}{\det(Id - tA)}.$$

Theorem

Assume $\zeta_{\sigma_A}(t) = \zeta_{\sigma_B}(t)$ *for two SFT's with adjacency matrices* A *and* B *respectively. Then,*

1. $h(X_A) = h(X_B)$.
2. $sp^{\times}(X_A) = sp^{\times}(X_B)$.

Proof.

We already know that zeta functions determine the set of periodic points and we also have $\limsup_{n\to\infty} \frac{1}{n} \log p_n(X) = h(X)$ for any sofic as well as SFT X.
The 2nd part follows from $\det(Id - tA) = \det(Id - tB)$. □

Right and Left Closing

Definition

A labeled graph is right-closing with delay D if whenever two paths of length $D+1$ start at the same vertex and have the same label, then they must have the same initial edge.

- Left closing is similarly defined and bi-closing is a left and right closing labeled graph.
- A labeled graph $\mathcal{G} = (G, \mathcal{L})$ is right-closing if and only if for each state $I \in \mathcal{V}(G)$, the map $\mathcal{L}^+$ is one-to-one on one sided paths starting at I.
- Suppose that $\mathcal{G} = (G, \mathcal{L})$, and that the 1-block code $\mathcal{L}_\infty : X_G \to X_\mathcal{G}$ is a conjugacy. Assume that $\mathcal{L}_\infty^{-1} = \Phi_\infty^{[-m,n]}$. Then, $\mathcal{L}$ is right closing with delay n.
- Delay of an out-splitting of a labeled graph $\mathcal{G}$ with delay D is $D+1$.
- For a right closing $\mathcal{G} = (G, \mathcal{L})$, $h(X_\mathcal{G}) = h(X_G)$.

Almost Finite Type

Definition

An irreducible sofic shift is called almost-finite-type (AFT) if it has a bi-closing presentation.

- The Fischer cover of any AFT is a bi-closing cover.
- Any other cover of an AFT intercepts its Fischer cover.
- Any irreducible sofic shift which has a bi-closing presentation, also has an almost invertible bi-closing presentation.

Coded Systems

Definition

(X, σ) is a **coded system** if it is the the closure of the set of sequences obtained by freely concatenating the words in $\mathcal{W}$, called the generator of X.

Other equivalent definitions are as follows.

- X has an irreducible right-resolving presentation.
- There are SFT's X_i s. t. $X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots$ and so that $X = \overline{\cup_{i=1}^{\infty} X_i}$.
- X has a **uniquely decipherable** generator.

Now we have the following inclusions

$$\cdots \subseteq \text{ sofics} \subseteq \text{ synchronized} \subseteq \text{ coded systems} \subseteq \text{ shift spaces.}$$

Half-synchronized Systems

It has been proved that

1. A sofic system is mixing iff it is totally transitive.
2. A sofic is mixing iff it has a generator $\mathcal{W}$ such that $\gcd(\mathcal{W}) = \gcd(\{|w_i| : w_i \in \mathcal{W}\}) = 1$.

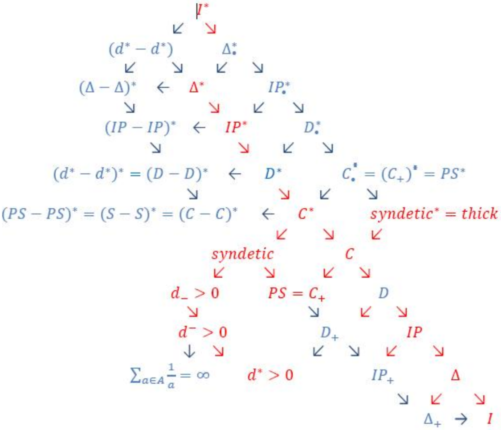
In fact, (1) is valid for a general coded system but (2) fails to be so.

Definition

A transitive subshift X is **half-synchronized** if there is $m \in \mathcal{B}(X)$, called the **half-synchronizing** word of X, and a left transitive ray $x_- \in X$ such that $x_{-|m|+1} \cdots x_0 = m$ and $F_X(x_-) = F_X(m)$.

- Half-synchronized shifts are coded.
- Any synchronized system is half-synchronized.
- Half-synchronized systems have a Fischer cover.

Other mixings and transitivities in topological and measure theoretical dynamical systems. Spacing shifts a good source of examples.



S-gaps

Definition

Let $S = \{s_i \in \mathbb{N} \cup \{0\} : 0 \leq s_i < s_{i+1},\, i \in \mathbb{N} \cup \{0\}\}$. Then the coded system over $\mathcal{A} = \{0,\, 1\}$ generated by

$$\mathcal{W}_S = \{10^{s_i} :\ s_i \in S\}$$

is called an S**-gap** shift and is denoted by $X(S)$.

- Any S-gap is synchronized. Any word containing 1 is synchronizing.
- $\mathcal{W}_S$ is uniquely decipherable.
- All S-gap shifts are **almost sofic**.

Properties of S-gaps

Set $\Delta(S) = \{d_n\}_n$ where $d_0 = s_0$ and $d_n = s_n - s_{n-1}$.

Theorem

An S-gap shift is

1. *SFT if and only if S is finite or cofinite;*
2. *An S-gap shift is AFT if and only if $\Delta(S)$ is eventually constant;*
3. *An S-gap shift is sofic if and only if $\Delta(S)$ is eventually periodic.*

Properties of S-gaps...

Theorem

Let S and $S^{'}$ be two different subsets of $\mathbb{N}_0$. Then $X(S)$ and $X(S^{'})$ are conjugate iff one of the S and $S^{'}$ is $\{0, n\}$ and the other $\{n, n+1, n+2, \ldots\}$ for some $n \in \mathbb{N}$.

Corollary

Suppose S and $S^{'}$ are two different non-empty subsets of $\mathbb{N}_0$. Then S and $S^{'}$ are conjugate iff they have the same zeta function.

Theorem

1. *The set of mixing S-gap shifts is an open dense subset of the space of S-gap shifts.*
2. *The set of non-mixing S-gaps is a Cantor dust (a nowhere dense perfect set).*

Properties of S-gaps...

Theorem

The map assigning to an S-gap shift its entropy is continuous.

Theorem

In the space of all S-gap shifts,

1. *The SFT S-gap shifts are dense.*
2. *The AFT S-gap shifts which are not SFT, are dense.*
3. *The sofic S-gap shifts which are not AFT, are dense.*
4. *An S-gap shift has specification with variable gap length if and only if $x_S \in \mathcal{B}$. (S-gap shifts having this property are uncountably dense with measure zero.) Here x_S is the real number assigned to $X(S)$ ($S \neq \{0, n\}$, $n \in \mathbb{N}$).*

β-shifts

Let $t \in \mathbb{R}$ and denote by $\lfloor t \rfloor$ the largest integer smaller than t. Let β be a real number greater than 1. Set

$$1_\beta = a_1 a_2 a_3 \cdots \in \{0, 1, \ldots, \lfloor \beta \rfloor\}^{\mathbb{N}},$$

where $a_1 = \lfloor \beta \rfloor$ and

$$a_i = \lfloor \beta^i (1 - a_1 \beta^{-1} - a_2 \beta^{-2} - \cdots - a_{i-1} \beta^{-i+1}) \rfloor$$

for $i \geq 2$. The sequence 1_β is the expansion of 1 in the base β; that is, $1 = \sum_{i=1}^{\infty} a_i \beta^{-i}$.

β-shift...

Let $\leq$ be the lexicographic ordering of $(\mathbb{N} \cup \{0\})^{\mathbb{N}}$. The sequence 1_β has the property that

$$\sigma^k 1_\beta \leq 1_\beta, \qquad k \in \mathbb{N},$$

where σ is the shift map.

Definition

(X_β, σ) where

$$X_\beta = \{x \in \{0, 1, \ldots, \lfloor \beta \rfloor\}^{\mathbb{Z}} : x_{[i, \infty)} \leq 1_\beta, \, i \in \mathbb{Z}\}.$$

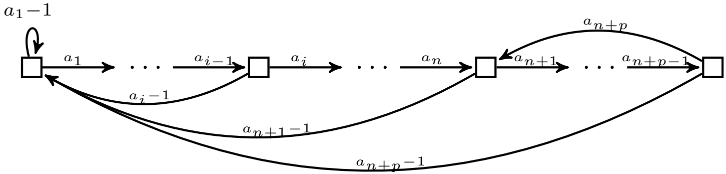
is called the β**-shift** over $\mathcal{A} = \{0, 1, \ldots, \lfloor \beta \rfloor\}$.

β-shift...

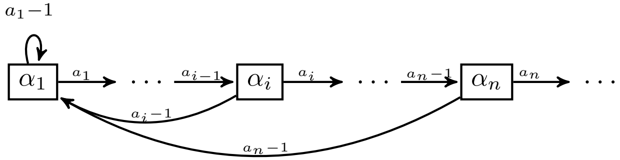
Theorem

The β-shift is

1. *half-synchronized.*
2. *is SFT iff the β-expansion of 1 is finite or purely periodic.*
3. *sofic iff the β-expansion of 1 is eventually periodic.*

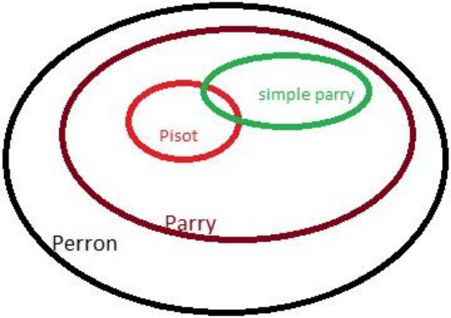


A typical Fischer cover of a strictly sofic β-shift for $1_\beta = a_1 a_2 \cdots a_n (a_{n+1} \cdots a_{n+p})^\infty$, $\beta \in (1, 2]$. The edges heading to the far left state exist if $a_i = 1$.



A typical Fischer cover of a nonsofic β-shift for $1_\beta = a_1 a_2 \cdots$, $\beta \in (1, 2]$. The edges ending at α_1 exist if $a_i = 1$.

β-shifts vs Number Theory



- β is simple Parry if 1_β is purely periodic.
- β is Parry if 1_β is eventually periodic.