Automorphism Groups of Low Complexity Minimal Subshifts

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Basic topological notion

Definition

Let (X, T) be a topological dynamical system, X a compact metric space. An automorphism $\phi\colon X \to X$ *is an homeomorphsim s.t.*

$$\phi \circ T = T \circ \phi.$$

$$\mathrm{Aut}(X, T) = \{\phi \text{ automorphism of } (X, T)\}.$$

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$$\langle T \rangle \subset \mathrm{Aut}(X, T)$$

Basic topological notion

Let A be a finite alphabet.
Let $X \subset A^{\mathbb{Z}}$ be a subshift invariant by the shift

$$\begin{aligned} \sigma : X &\to X \\ (x_n)_{n\in\mathbb{Z}} &\mapsto (x_{n+1})_{n\in\mathbb{Z}} \end{aligned}$$

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Theorem (Curtis-Hedlund-Lyndon)

Let ϕ be an automorphism of (X, σ)
There exists a local map $\hat{\phi}\colon A^{2r+1} \to A$ *s.t.*

$$\phi(x)_n = \hat{\phi}(x_{n-r} \dots x_{n+r}) \text{ for any } n \in \mathbb{Z}.$$

ϕ is a cellular automata.

Main theorem

The *complexity* $p_X\colon \mathbb{N} \to \mathbb{N}$, $p_X(n) = \sharp$ words of length n in X.

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Theorem (DDMP)

Let (X, σ) be a minimal subshift. If

$$\liminf_n \frac{p_X(n)}{n} < +\infty,$$

then $Aut(X, T)/\langle T \rangle$ is finite.

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Theorem (DDMP)

Let (X, σ) *be an aperiodic minimal subshift. If*

$$\liminf_n \frac{p_X(n)}{n} < +\infty,$$

then $Aut(X, T)/\langle T \rangle$ *is finite.*

Example. Primitive substitutive subshifts:
e.g. Tribonacci substitution

$$\tau(1) \mapsto 12, \ \tau(2) \mapsto 13, \ \text{and} \ \tau(3) \mapsto 1.$$

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Generalizes results of V. Salo-I. Törmä.
Similar result by V. Cyr-B. Kra

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Example. This includes also

- Subshifts of polynomial complexity of arbitrarily high degree.
- Subshifts with subexponential complexity $p_X(n) \geq g(n)$ i.o. where $\lim_n g(n)/\alpha^n = 0$ for any $\alpha \in \mathbb{R}$.

Previous results: in the measurable setting

Centralizer group: for a measurable dynamical system $(X, \mathcal{B}, \mu, T)$,

$$C(T) = \{\phi\colon X \to X;\ \text{bi-measurable},\ \phi \circ T = T \circ \phi\}$$

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- A. Del Junco (78): same is true for the Chacon subshift.
- J. King, J.-P. Thouvenot (91): mixing system of finite rank

$$C(T)/\langle T \rangle \text{ is finite.}$$

- G. A. Hedlund (69): For the Thue-Morse subshift, $\mathrm{Aut}(X, T)$ is generated by T and a flip map.

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- M. Hochman (2010): any SFT with positive entropy admits any finite group in $\mathrm{Aut}(X, T)$.

From the measurable to the topological setting

For zero-entropy system:

- B. Host, F. Parreau (89): for a family of substitutive systems

$$C(T) = \mathrm{Aut}(X, T) \text{ and } C(T)/\langle T\rangle \text{ is finite.}$$

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- M. Lemánczyk, M. Mentzen (89): any finite group can be realized as $C(T)/\langle T \rangle$.

Main Ideas

Lemma

Let (X, T) be a minimal aperiodic dynamical system. The action of $Aut(X, T)$ on X

$$\begin{array}{rcl} Aut(X, T) \times X & \rightarrow & X \\ (\phi, x) & \mapsto & \phi(x), \end{array}$$

is free.

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Proof. For any automorphism ϕ, the set

$$\{x; \phi(x) = x\}$$

is closed and T invariant.

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Two points $x, y \in (X, T)$ are *asymptotic* if

$$\lim_{n \to +\infty} \mathrm{dist}(T^n(x), T^n(y)) = 0.$$

Any infinite subshift admits an asymptotic pair.

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Any infinite subshift admits an asymptotic pair.

Any automorphism ϕ maps an asymptotic pair to an asymptotic pair

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Corollary

For an infinite t.d.s. (X, T), *with an asymptotic pair, we have*

$$\{1\} \longrightarrow \langle T \rangle \longrightarrow \ \ Aut(X,T) \ \ \stackrel{j}{\longrightarrow} \ \ Per\mathcal{A}_{/\sim},$$

where :

- $\mathcal{A}$ *denote the collection of asymptotic unordered pairs*
- $\{x, y\} \sim \{x', y'\}$ *if* x *and* x' *are in the same* T*-orbit.*
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If $\liminf_n P_X(n)/n < \infty$ then $\sharp\mathcal{A}_{/\sim} < +\infty$.

Main Ideas

Corollary

For an infinite subshift (X, σ), we have

$$\{1\} \longrightarrow \langle \sigma \rangle \longrightarrow \quad Aut(X, \sigma) \quad \xrightarrow{j} \quad Per\mathcal{A}_{/\sim},$$

where :

- *$\mathcal{A}$ denote the collection of asymptotic unordered pairs*
- *$\{x, y\} \sim \{x', y'\}$ if x and x' are in the same T-orbit.*
- *$Per\mathcal{A}_{/\sim}$ denotes the set of permutations on this set.*

If $\sharp\mathcal{A}_{/\sim} = 1$, then $Aut(X, T) = \langle T \rangle$.
e.g. for Sturmian sequences

More generally

In the same way: $x, y \in X$ are *proximal* if

$$\liminf_n dist(T^n x, T^n y) = 0.$$

$\phi \in \mathrm{Aut}(X, T)$ maps proximal points to proximal points.

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If (X, T) is a proximal extension of a minimal d-step nil system, then $Aut(X, T)$ is a d-step nilpotent group.

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Example. Toeplitz subshifts are proximal extension of their maximal equicontinuous factor ($d = 1$). Their automorphism group is Abelian.

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Question

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Relation with the complexity ?

Cyr and Kra: if $p_X(n)/n^2 \to 0$ then $\mathrm{Aut}(X, \sigma)$ is periodic