Problem Set 2

1. The *diagonal* Δ in $X \times X$ is the set of points of the form (x, x). Show that Δ is diffeomorphic to X, so Δ is a manifold if X is.

2. The *graph* of a map $f : X \longrightarrow Y$ is the subset of $X \times Y$ defined by

$$\operatorname{graph}(f) = \{(x, f(x)) : x \in X\}.$$

Define $F : X \longrightarrow \operatorname{graph}(f)$ by $F(x) = (x, f(x))$. Show that if f is smooth, F is a diffeomorphism; thus $\operatorname{graph}(f)$ is a manifold if X is. (Note that $\Delta = \operatorname{graph}(\text{identity})$.)

3. (a) Suppose that $f : X \longrightarrow Y$ is a smooth map, and let $F : X \longrightarrow X \times Y$ be $F(x) = (x, f(x))$. Show that

$$dF_x(v) = (v, df_x(v)).$$

   (b) Prove that the tangent space to $\operatorname{graph}(f)$ at the point $(x, f(x))$ is the graph of $df_x : T_x(X) \longrightarrow T_{f(x)}(Y)$.

4. A *curve* in a manifold X is a smooth map $t \longrightarrow c(t)$ of an interval of $\mathbb{R}^1$ into X. The *velocity vector* of the curve c at time t_0 -denoted simply $dc/dt\,(t_0)$- is defined to be the vector $dc_{t_0}(1) \in T_{x_0}(X)$, where $x_0 = c(t_0)$ and $dc_{t_0} : \mathbb{R}^1 \longrightarrow T_{x_0}(X)$. In case $X = \mathbb{R}^k$ and $c(t) = (c_1(t), \ldots, c_k(t))$ in coordinates, check that

$$\frac{dc}{dt}(t_0) = (c_1'(t), \ldots, c_k'(t)).$$

Prove that every vector in $T_x(X)$ is the velocity vector of some curve in X, and conversely. [HINT: It's easy if $X = \mathbb{R}^k$. Now parametrize.]

5. Prove that a local diffeomorphism $f : X \longrightarrow Y$ is actually a diffeomorphism of X onto an open subset of Y, provided that f is one-to-one.

6. *Generalization of the Inverse Function Theorem:* Let $f : X \longrightarrow Y$ be a smooth map that is one-to-one on a compact submanifold Z of X. Suppose that for all $x \in Z$,

$$df_x : T_x(X) \longrightarrow T_{f(x)}(Y)$$

is an isomorphism. Then f maps Z diffeomorphically onto $f(Z)$. (Why?) Prove that f, in fact, maps an open neighborhood of Z in X diffeomorphically onto an open neighborhood of $f(Z)$ in Y. Note that when Z is a single point, this specializes to the Inverse Function Theorem. [HINT: Prove that, by Exercise 5, you need only show f to be one-to-one on some neighborhood of Z. Now if f isn't so, construct sequences $\{a_i\}$ and $\{b_i\}$ in X both converging to a point $z \in Z$, with $a_i \neq b_i$ but $f(a) = f(b)$. Show that this contradicts the nonsingularity of df_z.]

7. (a) If X is compact and Y connected, show every submersion $f : X \longrightarrow Y$ is surjective.

   (b) Show that there exist no submersions of compact manifolds into Euclidean spaces.

8. *(Stack of Records Theorem.)* Suppose that y is a regular value of $f : X \longrightarrow Y$, where X is compact and has the same dimension as Y. Show that $f^{-1}(y)$ is a finite set $\{x_1, \ldots, x_N\}$. Prove there exists a neighborhood U of y in Y such that $f^{-1}(U)$ is a disjoint union $V_1 \cup \cdots \cup V_N$ where V_i is an open neighborhood of x_i and f maps each V_i diffeomorphically onto U. [HINT: Pick disjoint neighborhoods W_i of x_i that are mapped diffeomorphically. Show that $f(X - \cup W_i)$ is compact and does not contain y.]

9. Let X and Z be transversal submanifolds of Y. Prove that if $y \in X \cap Z$, then

$$T_y(X \cap Z) = T_y(X) \cap T_y(Z).$$

("The tangent space to the intersection is the intersection of the tangent spaces.")

10. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a sequence of smooth maps of manifolds, and assume that g is transversal to a submanifold W of Z. Show $f \pitchfork g^{-1}(W)$ if and only if $g \circ f \pitchfork W$.

11. Let $f : X \longrightarrow X$ be a map with fixed point x; that is, $f(x) = x$. If $+1$ is not an eigenvalue of $df_x : T_x(X) \longrightarrow T_x(X)$, then x is called a *Lefschetz fixed point of* f. f is called a *Lefschetz map* if all its fixed points are Lefschetz. Prove that if X is compact and f is Lefschetz, then f has only finitely many fixed points.

12. Show that the *antipodal map* $x \longrightarrow -x$ of $S^k \longrightarrow S^k$ is homotopic to the identity if k is odd. HINT: Start off with $k = l$ by using the linear maps defined by

$$\begin{pmatrix} \cos \pi t & -\sin \pi t \\ \sin \pi t & \cos \pi t \end{pmatrix}.$$

13. Prove that diffeomorphisms constitute a stable class of mappings of compact manifolds; that is, prove part (f) of the Stability Theorem. [HINT: Reduce to the connected case. Then use the fact that local diffeomorphisms map open sets into open sets, plus part (e) of the theorem.

    Stability theorem. The following classes of smooth maps of a compact manifold X into a manifold Y are stable classes:

    (a) local diffeomorphisms.

    (b) immersions.

    (c) submersions.

    (d) maps transversal to any specified submanifold $Z \subset Y$.

    (e) embeddings.

    (f) diffeomorphisms.]

14. Prove that the Stability Theorem is false on noncompact domains. Here's one counterexample, but find others yourself to understand what goes wrong. Let $\rho : \mathbb{R} \longrightarrow \mathbb{R}$ be a function with $\rho(s) = 1$ if $|s| < 1$, $\rho(s) = 0$ if $|s| > 2$. Define $f_t : \mathbb{R} \longrightarrow \mathbb{R}$ by $f_t(x) = x\rho(tx)$. Verify that this is a counterexample to all six parts of the Stability theorem. [For part (4), use $Z = \{0\}$.]