Introduction to Group Actions

2nd Problem Set
Due Aban 9th, 1398

- Exercise 2.1.2, page 24 from [N].

- Exercises 7.1.1, 7.1.2 and 7.1.3, page 159 from [BS].

- Let $\{a_n\}_n$ be a sequence of real numbers, so that there is some contstant $C > 0$ satisfying

$$|a_{m+n} - a_m - a_n| \leq C, \quad \text{for all } m, n \in \mathbb{N}.$$

  a) Show that the limit $\lim_{n\to\infty} \frac{a_n}{n}$ exists.
  b) If we denote the value of this limit by ρ, prove that the sequence $\{a_n - n\rho\}_n$ is bounded.

- Let $\mathbb{H}$ denote the upper half plane in $\mathbb{C}$,

$$\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}.$$

  For any matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}(2, \mathbb{R})$ define $f_A(z) = \frac{az+b}{cz+d}$.
  a) Show that for $A \in \text{SL}(2, \mathbb{R})$, f_A defines a homeomorphism of $\mathbb{H}$.
  b) Show that $f_A = f_B$ if and only if $A = \pm B$.
  c) Show that for any pair of hyperbolic matrices A, B in $\text{SL}(2, \mathbb{R})$, two homeomorphisms f_A and f_B are conjugate.
  d) Show that for any pair of parabolic matrices A, B in $\text{SL}(2, \mathbb{R})$, two homeomorphisms f_A and f_B are conjugate.
  e) Show that the statement of part (c) and (d) is not true for elliptic matrices.

- Let $R : \mathbb{S}^1 \to \mathbb{S}^1$ be an irrational rotation on the circle. Show that the centralizer of R in $\text{Homeo}^+(\mathbb{S}^1)$ is exactly the set of rotations, i.e. if an orientation preserving homeomorphism g commutes with R, then g is a rotation. Is the same statement true for rational rotations?

- a) Show that a homeomorphism $f \in \text{Homeo}^+(\mathbb{S}^1)$ does not admit both finite and dense orbit.
  b) Give an example of two homeomorphisms $f, g \in \text{Homeo}^+(\mathbb{S}^1)$ so that the the action of the group G generated by f, g on $\mathbb{S}^1$ admits

both finite and dense orbits.

- Let S_n be the group of permutations on n elements.
  a) Show that for any $n \geq 3$, $\mathrm{Homeo}^+(\mathbb{S}^1)$ does not contain a subgroup isomorphic to S_n.
  b) Show that for any $n \geq 4$, $\mathrm{Homeo}(\mathbb{S}^1)$ does not contain a subgroup isomorphic to S_n.

References

[BS] Brin, M., Stuck, G., *Introduction to Dynamical Systems*, Cambridge University Press.

[N] Navas, A., *Groups of Circle Diffeomorphisms*, The University of Chicago Press.

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