Neural Oscillators: Weak Coupling

Part III:

Who Cares About Adjoints?

Adjoint and Response to Inputs, Forced Oscillators

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Who Cares About Adjoints?

We have talked a lot about PTCs, PRCs, STRCs and Adjoints, but **Why?**

There are *at least two ways* in which these response functions are useful:

- We can use them directly to **create systems of pulse-coupled equations** which we can reduce to **maps**, or
- We can use the **weak coupling** assumption and averaging to reduce arbitrarily coupled networks of neurons to **systems of equations on a torus**.

Adjoint and the Response to Inputs

The adjoint and its cousin the PRC are relevant beyond purely oscillating neurons. They provide information about how a neuron responds to inputs **even when it is not a regular oscillator**.

If the neuron is in a **stable resting state**, then a *weak input* will have *no effect* on it and only strong inputs will produce a spike. So, there is no dependence of the response on the time of the inputs since the cell is at rest.

If the neuron is subject to sufficient depolarization so that it is **firing** (irregularly, perhaps), then even **weak inputs can alter the time of a spike.** *The amount by which the spike time is altered depends on when the neuron last spiked and thus we can expect that there is some relationship to the PRC.*

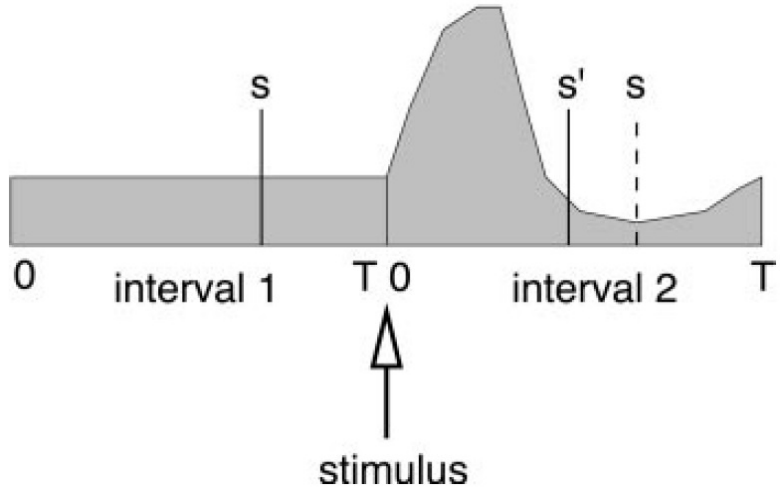
B. S. Gutkin, G. B. Ermentrout, and A. D. Reyes. *Phase-response curves give the responses of neurons to transient inputs.* J. Neurophysiol., 94:1623-1635, 2005.

They have **numerically** shown that the **shifting of spikes** in a **nonstationary system** is determined by the **steady-state PRC** of the neuron. *That is, the PRC can tell us how an input shifts a spike even when the neuron is not a regular oscillator.*

Post-Stimulus Time Histogram (PSTH)

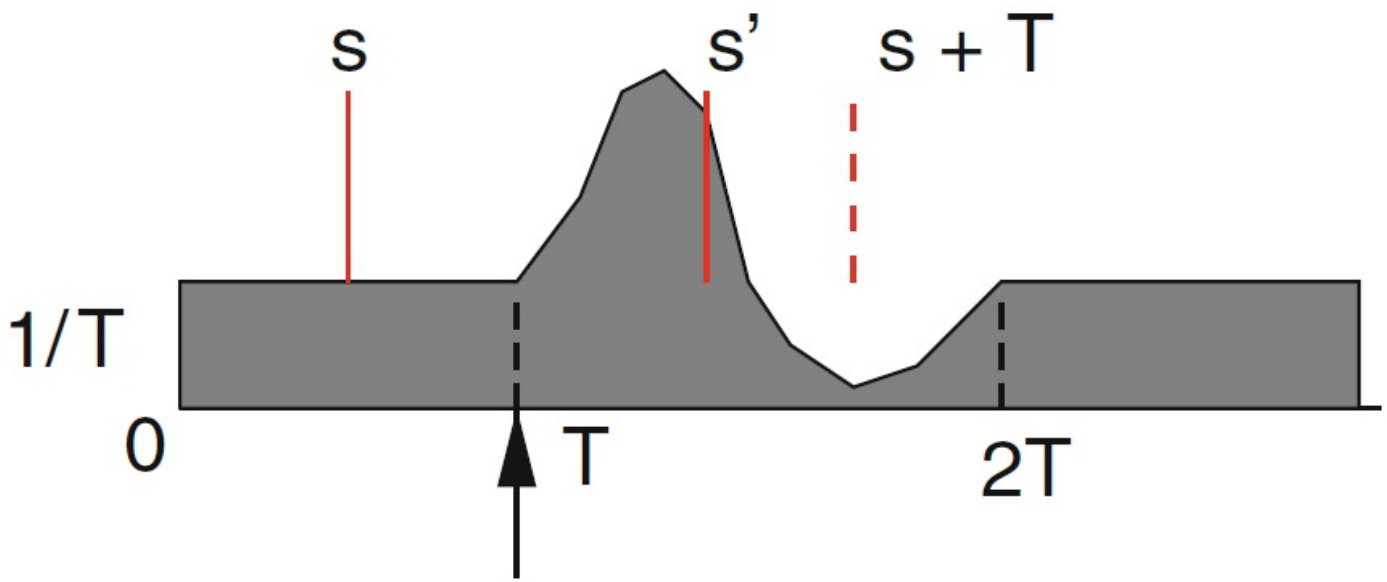
Post-Stimulus Time Histogram (PSTH): The probability of a spike occurring at a given time t after the onset of a stimulus.

It is measured by collecting the spike times of a neuron over many **repetitions** of the same stimulus. Implicit in the usefulness of the PSTH is that the dynamics are **stationary**. For example, the mean firing rate of the neuron before the stimulus is constant and the time between repetitions is long enough so that the effect of the stimulus wears off.



We assume:

- 1. The **mean firing rate** of the neuron is 1/T prior to the stimulus and the probability over trials of firing is **uniform**.
- 2. The stimulus is sufficiently weak so that **no new spikes are added.** That is, the probability of firing is shifted, but there are no new spikes.
- 3. The neuron rapidly returns to its baseline mean firing rate after one cycle.



Let

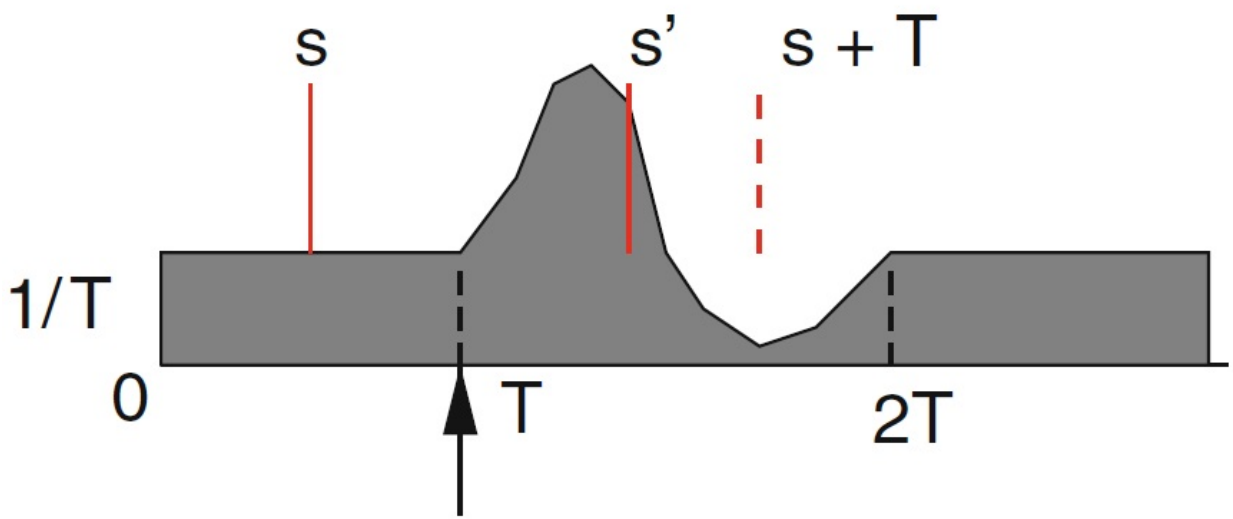
- s: previous spike time,
- T: the time of onset of stimulus,
- s': next spike time,

Therefore

$$s' = s + T + \Delta(T - s) \equiv F(s)$$

Since stimulus is assumed to be weak, we have

$$F'(s) = 1 - \Delta'(T - s) > 0.$$



As $F(0) = T$ and $F(T) = 2T$, the map $F : [0, T] \to [T, 2T]$ is invertible (increasing). From definition of *PSTH*, we have:

$$\mathbb{P}\{s' < t\} = \int_T^t PSTH(t')\mathrm{d}t'.$$

Also, we have

$$\begin{aligned} \mathbb{P}\{s' < t\} &= \mathbb{P}\{F(s) < t\} \\ &= \mathbb{P}\{s < F^{-1}(t)\} \\ &= \tfrac{1}{T}F^{-1}(t) \end{aligned}$$

So

$$F^{-1}(t) = T\int_T^t PSTH(t')\mathrm{d}t'$$

which provides us with the relation between *PSTH* and *PRC*.

Forced Oscillators

PRCs are **directly computable from experimental data**; thus, they provide a way of creating a model of a specific biological oscillator without needing a mechanistic basis.

Question: How one can use the **PRC** to develop a **map** for the dynamics of a single oscillator which is **periodically forced by an external pulsatile stimulus**?

L. Glass and M. C.Mackey.*From Clocks to Chaos: The Rhythms of Life*. Princeton University Press, Princeton, NJ, 1988.
R. Stoop, L. A. Bunimovich, and W. H. Steeb. *Generic origins of irregular spiking in neocortical networks.* Biol. Cybern., 83:481-489, 2000.
R. Stoop, K. Schindler, and L. A. Bunimovich. *When pyramidal neurons lock, when they respond chaotically, and when they like to synchronize.* Neurosci. Res., 36:8191, 2000.

Now let that

- $X(t)$ be an oscillator,
- $\Delta(t)$ be the PRC of the oscillator which describes the advance or delay as a function of the time of the stimulus, $0 \leq t < T$,
- T be the period of oscillator,
- T_f be the period of the stimulus arrivals(periodic),
- $\{\phi_n\}_n$ be the sequence of phases of the oscillator at the instants exactly before the stimula arrivals,

Therefore we have

$$\phi_{n+1} = \phi_n + \Delta(\phi_n) + T_f = P(\phi_n) + T_f = M(\phi_n),$$

where

$$M(\phi) \equiv P(\phi) + T_f = \phi + \Delta(\phi) + T_f.$$

Since *PRC* theory is valid mainly when stimuli are weak, the size of Δ is assumed to be small. Therefore, $P(\phi)$ is monotonic and its derivative is $M'(\phi) = P'(\phi) = 1 + \Delta'(\phi)$ is nonzeroe.

Homeomorphism of Circle and Rotation Number

Suppose that $f : \mathbb{S}^1 \to \mathbb{S}^1$ is an orientation preserving homeomorphism of the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. Then f may be lifted to a homeomorphism $F : \mathbb{R} \to \mathbb{R}$ of the real line, satisfying

$$F(x + m) = F(x) + m$$

for every real number x and every integer m.

The **rotation number** of f is defined in terms of the iterates of F:

$$\omega(f) \equiv \lim_{n \to \infty} \frac{F^n(x) - x}{n}.$$

Poincar proved that the limit exists and is independent of the choice of the starting point x. The lift F is unique modulo integers, therefore the rotation number is a well-defined element of $\mathbb{R}/\mathbb{Z}$. Intuitively, it measures the **average rotation angle along the orbits of** f.

Homeomorphism of Circle and Rotation Number

Example: If f is a rotation by θ, so that

$$F(x) = x + \theta,$$

then its rotation number is θ.

The rotation number is invariant under topological conjugacy, and even topological **semiconjugacy**: if f and g are two homeomorphisms of the circle and

$$h \circ f = g \circ h$$

for a continuous map h of the circle into itself (not necessarily homeomorphic) then f and g have the same rotation numbers. It was used by Poincare and Arnaud Denjoy for topological classification of homeomorphisms of the circle.

Rotation Number and Denjoy Theorem

There are two distinct possibilities.

Case I: The rotation number of f is a rational number p/q (in the lowest terms). Then f has a periodic orbit, every periodic orbit has period q, and the order of the points on each such orbit coincides with the order of the points for a rotation by p/q. Moreover, every forward orbit of f converges to a periodic orbit. The same is true for backward orbits, corresponding to iterations of f^{-1}, but the limiting periodic orbits in forward and backward directions may be different.

Rotation Number and Denjoy Theorem

Case II: The rotation number of f is an irrational number θ. Then f has no periodic orbits There are two subcases.

- There exists a dense orbit. In this case f is topologically conjugate to the irrational rotation by the angle θ and all orbits are dense. Denjoy proved that this possibility is always realized when f is twice continuously differentiable.
- There exists a Cantor set C invariant under f. Then C is a unique minimal set and the orbits of all points both in forward and backward direction converge to C. In this case, f is semiconjugate to the irrational rotation by θ, and the semiconjugating map h of degree 1 is constant on components of the complement of C.

Rotation number is continuous when viewed as a map from the group of homeomorphisms (with C^0 topology) of the circle into the circle.

Discussion for M

Similarly, for M we define

$$\rho \equiv \lim_{n\to\infty} \frac{1}{T}\frac{\phi_n}{n}.$$

If $\rho = p/q$, then we have a phase locking which we call it $p : q$ locking.

Theorem

The rotation number is well defined; i.e. the limit exists and is independent of the initial condition. Furthermore, if $M \in C^2$, then: **(i)** *ρ is rational if and only if $M(\phi)$ has a periodic orbit of some period N, i.e. $\phi_{n+N} = \phi_n$ mod T.* **(ii)** *ρ is irrational if and only if every orbit $\{\phi_n\}$ is dense in the circle.* **(iii)** *ρ is a continuous function of any parameters in the function M.*

Discussion for M

$1:1$ **locking**:
Roughly speaking, we must have $\phi_{n+1} = \phi_n + T$; that is, the oscillator must complete one cycle per stimulus. This means

$$\phi + T = \phi + \Delta(\phi) + T_f$$

so we must solve $\Delta(\phi) = T - T_f$ for ϕ.

In **class I** neurons, the *PRC* is typically nonnegative everywhere, so to entrain this type of neural oscillator, the period of the forcing stimulus must be smaller than the natural period.

On the other hand, for **class II** firing, the *PRC* can be both positive and negative, so the period of the forcing stimulus can be both shorter and longer than the natural period.

Discussion for M

To determine the stability, we linearize

$$M(\phi) = \phi + \Delta(\phi) + T_f$$

for ϕ_n and obtain

$$y_{n+1} = (1 + \Delta'(\phi_n)) \equiv cy_n$$

where $c > 0$ since the PTC is monotonic.
We have:

- If $\Delta'(\phi_n) > 0$, then $c > 1$ the fixed point is unstable.
- If $\Delta'(\phi_n) < 0$, then $0 < c < 1$ the fixed point is stable.
- The only bifurcation that can occur corresponds to $c = 1$ or $\Delta'(\phi_n) = 0$, which means ϕ is a double root and occurs at a local maximum or minimum of $\Delta(\phi)$.

Discussion for M

$1:2$ **locking**:
Roughly speaking, it means the oscillator fires only once for every two stimuli and we must have $\phi_{n+2} = \phi_n + T$; As we have $\phi_{n+2} = M(\phi_{n+1}) = M(M(\phi_n))$, we must solve

$$T = 2T_f + \Delta(\phi) + \Delta[\phi + \Delta(\phi) + T_f]$$

for ϕ. Obviously, this is more difficult to solve than the $m:1$ case.

However, we note that if Δ is small, then this can be solved only when $2Tf \approx T$. That is, the stimulus must be roughly twice as fast as the intrinsic period of the oscillator, another intuitively appealing observation.

Discussion for M

Stability proceeds in a similar manner and we require that $c = M'(M(\phi))M'(\phi) < 1$ for stability. This gives a condition on $\Delta(\phi)$ and its derivatives:

$$c = [1 + \Delta'(\phi)][1 + \Delta'(\phi + \Delta(phi) + T_f)] < 1.$$

In general, for $m : r$ **locking**, in which the stimulus fires r times and the oscillator fires m times, we have to solve $\phi + mT = M^r(\phi)$. Note that

$$M^r(\phi) = \phi + rT_f + ...$$

where the remaining terms depend on Δ. Thus, for small *PRC*s, we want $T_f/T \approx m/r$.

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Discussion for M

The analysis above hints that whenever T_f/T is near a rational number, m/r, then there can be phase-locking and thus a rational rotation number.

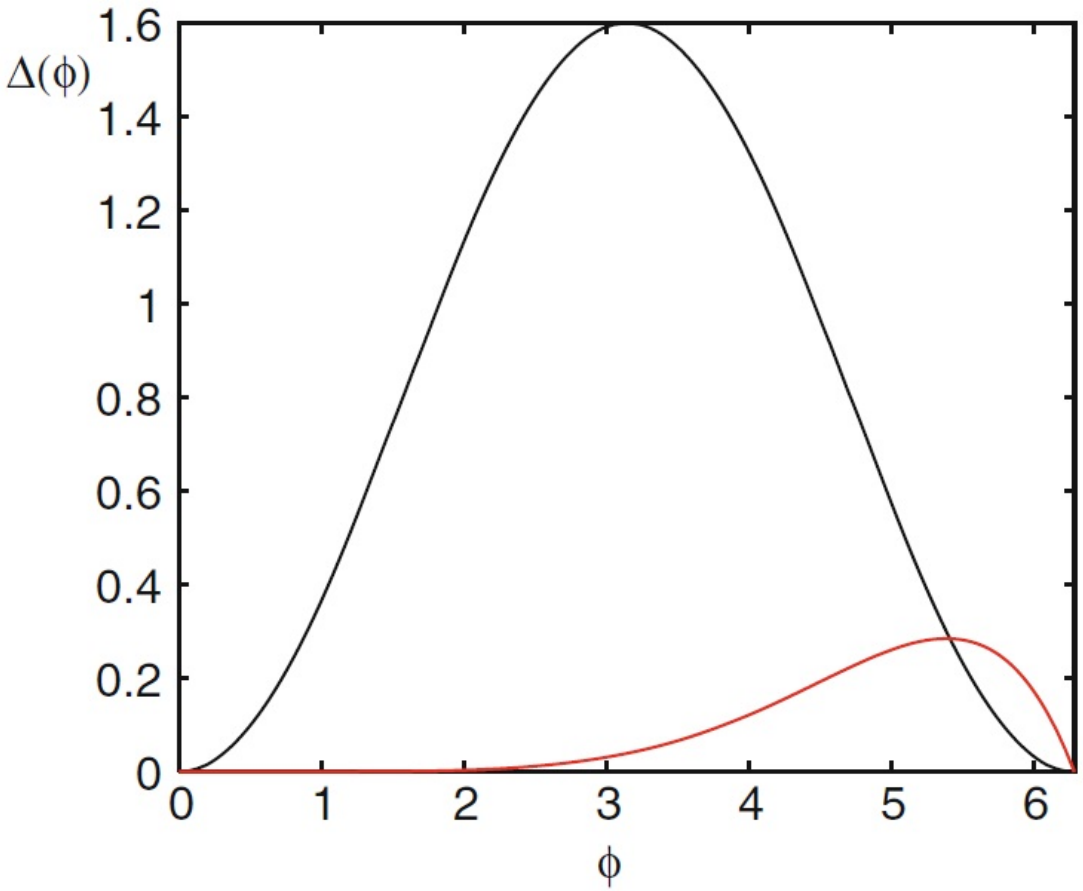
If the *PRC* is nontrivial (that is, it is not constant), then we should be able to solve

$$\phi + mT = M^r(\phi)$$

for ϕ and thus find m/r locked solutions for some open set of forcing periods, T_f. This means if T_f is a parameter, then $\rho(T_f)$ should be constant over the open regions where there is locking.

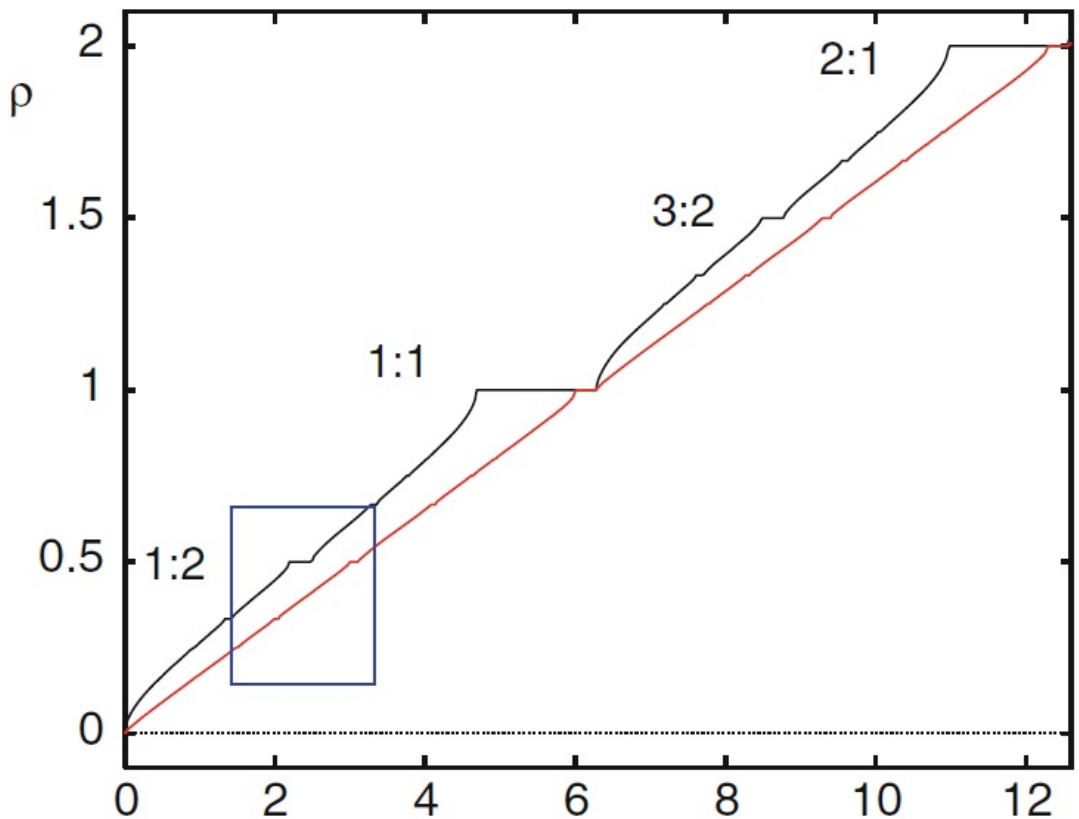
Discussion for M

Let $\Delta_1(\phi) = 0.8(1 - \cos(\phi))$ and $\Delta_2(\phi) = 0.000013\phi^6(2\pi - \phi)$.



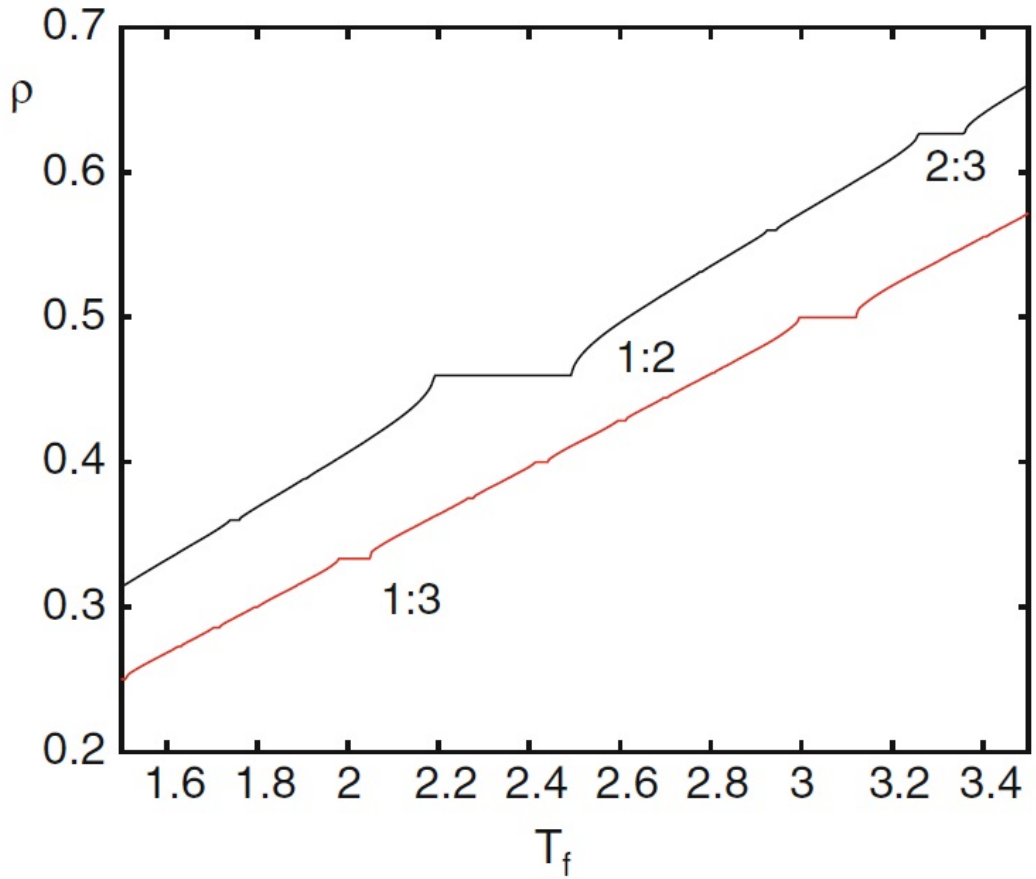
Devils staircase

Since the magnitude of Δ_2 is much smaller than that of Δ_1 , the width of the plateaus is much shorter. In both cases, the rotation number increases with T_f and is punctuated with regions where it is constant.



Devils staircase

Pattern in steps: Between any flat regions $a : b$ and $c : d$, the largest regime is always the one obeying the Farey addition rule, i.e $(a + c) : (b + d)$.



Devils staircase

If the slope of the *PRC* becomes more negative than -1, the invertibility of M is violated and it is possible to get very complex behavior in these simple maps, such as chaos. In neural models, a slope of -1 means the neuron fires at the instant of the stimulus. Thus, it is impossible to get slopes more negative than -1.

Thanks for your patience!